SAMP: A Semi-Analytical Model for the Simulation of Polymers

S. Kolling a, A. Haufe b, M. Feucht a & P.A. Du Bois c

a DaimlerChrysler AG, EP/SPB, HPC X411, 71059 Sindelfingen, Germany
b DYNAMore GmbH, Industriestr. 2, 70565 Stuttgart, Germany
c Consulting Engineer, Freiligrathstr. 6, 63071 Offenbach, Germany

Correspondence:
Dr. Stefan Kolling
DaimlerChrysler AG
HPC X411
D-71059 Sindelfingen
Germany
Tel: +49-(0)7031 - 9082829
Fax: +49-(0) 7031 - 9078837
e-mail: stefan.kolling@daimlerchrysler.com
1 Introduction

The numerical simulation of structural parts made from plastics is becoming increasingly important nowadays. The fact that almost any structural requirement can be combined in a lightweight, durable and cost effective structure is the driving force behind its widespread application. More and more structural relevant parts are being constructed and manufactured from plastics. This on the other hand drives the demand for reliable and robust methods to design these parts and to predict their structural behaviour. The key ingredients that need to be available are verified, calibrated and validated constitutive models for any family of plastic material. This holds not only true for crashworthiness applications but for any other application field.

Under high velocity impact loading, thermoplastic components undergo large plastic deformations and will most likely fail. Consequently, the unloading behaviour is irrelevant and thermoplastics can be modelled with a pretty good approximation as pseudo-metallic elastic-plastic bodies. This is, however, not always the case – even in crashworthiness applications. Nowadays important applications in crash simulation that demand a more accurate modelling of thermoplastics are simulations in pedestrian protection, e.g. head and leg impact (see Error! Reference source not found.–[15]), (??? Stefan: Referenzen sind nicht verlinkt ???) and passenger protection. Although highly sophisticated material laws are available in commercial finite element programs, there are still open questions, especially in the aforementioned field of application. In this paper the main focus is set to the explicit solver of LS-DYNA Error! Reference source not found.. Error! Reference source not found., but clearly with some effort any results are transferable to other solvers. In the following an overview on classical models for polymers which are used for crash simulations nowadays is given. From a practical point of view, the usually applied constitutive model is material #24 (MAT_PIECEWISE_LINEAR_PLASTICITY), a classical elastic-plastic model based on the vonMises criteria. It should be strongly emphasised though, that thermoplastics are not incompressible during plastic flow. This leads to the conclusion that material laws based on the vonMises criterion are not suitable in general. Therefore, a new material model which has been implemented into LS-DYNA as a user defined constitutive model will be the main focus of the present contribution.

Based on experimental work, important phenomena like necking, strain rate dependency, unloading behaviour and damage are identified for certain polymer materials. Subsequently, a constitutive model including the experimental findings and phenomena is derived. In particular, different behaviour in compression, tension and shear, as well as a strong strain rate dependent failure need to be addressed. Also, strain dependent damaging and damage induced erosion are both noticeable properties that need to be included in a constitutive model. The necessity of a pressure dependent formulation has been shown in [5] where a classical Drucker-Prager [8 & 9] formulation was used to simulate a simple three point bending test. But for the aforementioned application in pedestrian protection, i. e. leg impact, this simple model is not accurate enough. In such load cases the bumper fascia will typically undergo only small straining and the deflection will be largely elastic while the unloading phase is of fundamental importance for the determination of the bending angle in the leg-form. Here the degradation of elastic parameters (damaging) and the visco-elastic response is of uttermost importance. These key properties will be exemplified in the following discussion:

The true stress-true strain curve of a polymer is given in Error! Reference source not found.. A reversal of curvature indicating a softening phase that is followed by a hardening phase can be identified. Thus a typical bone shaped specimen in a tension test shows necking at very low strains corresponding to the initial softening of the material. The subsequent hardening, however, results in a stabilization of the necked area and a redistribution of the plastic strains over the entire specimen. Unlike metals, the primary energy-absorbing potential of the thermoplastic resides at plastic strains beyond the necking value.
Figure 1: True stress - true strain curves of PP-EPDM

This type of physical response can be modelled perfect using standard elastic-plastic material laws although the plastic deformation of thermoplastics is not isochoric as mentioned above. A further effect which has to be considered, though, is the unloading behaviour of the material. Usually, this is addressed by a damage parameter. It will be shown, that an approach where a scalar damage value is tabulated against equivalent plastic strains is a simple yet effective means to model such behaviour.

In conclusion it can be said that all the effects associated with thermoplastics can be approximately considered in simple material models: Necking by an elastic-plastic law, unloading behaviour by a damage model; pressure dependent behaviour by standard Drucker-Prager model. A new constitutive model, termed as SAMP-1 (Semi-Analytical Model for Polymers with C\(^1\)-differentiable yield surface) is derived in the following. It was the main focus of the present project to include all relevant experimentally observed effects in this formulation. Hence, the proposed model was expected and it was found to deliver extraordinary results in all available verification tests. Even applications to materials that were not targeted at the beginning is possible with the greatest success.

2 Material law formulation

2.1 Yield surface formulation for plastics

Elastic-plastic material laws have been developed historically for the description of metallic materials based on crystal plasticity. The most commonly used example of this type implemented in LS-DYNA is MAT_PIECEWISE_LINEAR_PLASTICITY. The same approach can be applied to some degree to the simulation of thermoplastics. However, it should be noted that very important differences exist between metals and thermoplastics. In particular, plastics have no constant modulus of elasticity and the different yield criterions under tension and compression preclude the use of a von Mises type of yield surface. Furthermore, the hardening of thermoplastics is anisotropic and the plastic deformation does not happen at constant volume. This lack of plastic incompressibility requires a flow rule allowing for permanent volumetric deformation. None of these effects can be considered in a classical metallic elastic-plastic material law. For an overview, some materials considering plasticity limited to materials with isotropic behavior are listed in Table 1.

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<td>15</td>
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<td>JC</td>
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<td>MAT_SIMPLIFIED_JOHNSON_COOK</td>
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<td>Load curve</td>
<td>CS, tabulated</td>
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2.1.1 Choice of a yield surface formulation

All plastics are to some degree anisotropic. The anisotropic characteristic can be due to fibre reinforcement, to the moulding process or it can be load induced in which case the material is at least initially isotropic. Therefore a quadratic form in the stress tensor is often used to describe the yield surface (see Bardenheier, Feng…). We restrict the scope of this work to isotropic formulations. However, the choice of a Tsay-Wu yield surface was made in view of later anisotropic generalisations. In the isotropic case the most general quadratic yield surface can be written as

\[ f = \sigma^T F \sigma + B \sigma + F_0 \leq 0, \tag{1} \]

where

\[
\sigma = \begin{pmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{zz} \\
\sigma_{xy}
\end{pmatrix}, \quad F = \begin{pmatrix}
F_{11} & F_{12} & F_{12} & 0 & 0 & 0 \\
F_{12} & F_{11} & F_{12} & 0 & 0 & 0 \\
F_{12} & F_{12} & F_{11} & 0 & 0 & 0 \\
0 & 0 & 0 & F_{44} & 0 & 0 \\
0 & 0 & 0 & 0 & F_{44} & 0 \\
0 & 0 & 0 & 0 & 0 & F_{44}
\end{pmatrix}, \quad B = \begin{pmatrix}
F_1 & 0 & 0 & 0 & 0 & 0 \\
0 & F_1 & 0 & 0 & 0 & 0 \\
0 & 0 & F_1 & 0 & 0 & 0 \\
0 & 0 & 0 & F_1 & 0 & 0 \\
0 & 0 & 0 & 0 & F_1 & 0 \\
0 & 0 & 0 & 0 & 0 & F_1
\end{pmatrix} \tag{2}
\]

Some restrictions apply to the choice of the coefficients. The existence of a stress-free state and the equivalence of pure shear and biaxial tension/compression require respectively

\[ F_0 \leq 0 \quad \text{and} \quad F_{44} = 2(F_{11} - F_{12}) \tag{3} \]

Although 4 independent coefficients remain in the expression for the isotropic yield surface at this point, however the yield condition is not affected if all coefficients are multiplied by a constant. Consequently only 3 coefficients can be freely chosen and 3 experiments under different states of stress can be fitted by this formulation.

Without loss of generality the expression for the yield surface can be reformulated in terms of the first two stress invariants: pressure and vonMises stress:

\[
p = -\frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3},
\]

\[
\sigma_{\text{vm}} = \sqrt{\frac{3}{2} \left( (\sigma_{xx} + p)^2 + (\sigma_{yy} + p)^2 + (\sigma_{zz} + p)^2 + 2\sigma_{xy}^2 + 2\sigma_{yz}^2 + 2\sigma_{zx}^2 \right)} \tag{4}
\]

The expression for the yield surface then becomes

\[ f = \sigma_{\text{vm}}^2 - A_0 - A_1 p - A_2 p^2 \leq 0 \tag{5} \]

and identification of the coefficients gives

\[ A_0 = -F_0, \quad A_1 = 3F_1 \quad \text{and} \quad A_2 = 9(1 - F_{11}) \tag{6} \]

or equivalently
\[ F_0 = -A_0, \quad F_1 = \frac{A_1}{3}, \quad F_{11} = 1 - \frac{A_2}{9}, \quad F_{44} = 3 \quad \text{and} \quad F_{12} = F_{11} - \frac{F_{44}}{2} = -\left(\frac{1}{2} + \frac{A_2}{9}\right). \] (7)

Since there is no loss of generality, the simpler formulation in invariants is adopted from this point on. In principle the coefficients of the yield surface can now be determined from 3 experiments. Typically we would perform uniaxial tension, uniaxial compression and simple shear tests:

\[ \sigma_x, \quad \sigma_y, \quad \sigma_y \quad \text{and} \quad \sigma_y \quad \text{and} \quad \sigma_y \]

\[ \begin{align*}
3\sigma_r^2 &= A_0 \\
\sigma_r^2 &= 3\sigma_s^2 - A_1\sigma_1^2 + A_2\sigma_2^2 \\
\sigma_r^2 &= 3\sigma_s^2 + A_1\sigma_1^2 + A_2\sigma_2^2
\end{align*} \]

\[ \Rightarrow \begin{cases} 
A_0 = 3\sigma_s^2 \\
A_1 = 9\sigma_s^2\left(\frac{\sigma_1 - \sigma_r}{\sigma_r}\right) \\
A_2 = 9\left(\frac{\sigma_s - 3\sigma_s^2}{\sigma_s\sigma_r}\right)
\end{cases} \] (8)

Alternatively we can also compute the coefficients relating to the formulation in stress space:

\[ \begin{align*}
F_0 + F_1\sigma_r + F_{11}\sigma_r^2 &= 0 \\
F_0 - F_1\sigma_r + F_{11}\sigma_r^2 &= 0 \\
F_0 + F_{44}\sigma_s^2 &= 0
\end{align*} \]

\[ \Rightarrow \begin{cases} 
F_1 = F_0\left(\frac{1}{\sigma_r} - \frac{1}{\sigma_r}\right) \\
F_{11} = -\frac{F_0}{\sigma_s\sigma_r} \\
F_{44} = -\frac{F_0}{\sigma_s^2}
\end{cases} \] (9)

Both are easily seen to be equivalent.

2.1.2 Conditions for convexity of the yield surface

2.1.2.1 Introductory remarks

Usually the yield surface is required to be convex, i.e.

\[ \begin{align*}
f(\sigma_1) &\leq 0 \\
f(\sigma_2) &\leq 0 \\
0 &\leq \alpha \leq 1
\end{align*} \Rightarrow f(\alpha\sigma_1 + (1 - \alpha)\sigma_2) \leq 0 \] (10)

The second derivative of \( f \) is computed as
\[ f = \sigma^T F \sigma + B \sigma + F_0 \rightarrow \frac{\partial^2 f}{\partial \sigma^2} = 2F \] (11)

A sufficient condition for convexity in 6D stress space is then that the matrix \( F \) should be positive semidefinite. This means all eigenvalues of \( F \) should be positive or zero. The conditions for convexity will now be examined in physical terms for two cases: plane stress and general 3D.

2.1.2.2 The plane stress case

In the plane stress case the yield condition reduces to:

\[ f = \sigma^T F \sigma + B \sigma + F_0 \] (12)

where

\[
\begin{bmatrix}
\sigma_{xx} \\
\sigma_{yy} \\
\sigma_{xy}
\end{bmatrix}, \quad
F = \begin{bmatrix}
F_{11} & F_{12} & 0 \\
F_{12} & F_{11} & 0 \\
0 & 0 & F_{44}
\end{bmatrix}, \quad
B = \begin{bmatrix}
F_1 & 0 & 0 \\
0 & F_1 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\] (13)

And convexity requires the eigenvalues of \( F \) to be non-negative:

\[
\begin{align*}
F_{11} + F_{12} & \geq 0 \\
F_{11} - F_{12} & \geq 0 \\
F_{44} & \geq 0
\end{align*}
\] (14)

Representation of the yield surface in the principal stress plane allows a geometrical interpretation, see Figure 2.

![Figure 3: Representation of the yield surface in the principal stress plane](image)

Clearly as long as

\[ \sigma_s > \frac{\sqrt{\sigma_l \sigma_u}}{2}, \] (15)

the yield surface is elliptic and thus convex. In the case
\[ \sigma_s = \frac{\sqrt{\sigma_1 \sigma_2}}{2} \]  

(16)  

the yield surface degenerates into two straight lines. And for  

\[ \sigma_s < \frac{\sqrt{\sigma_1 \sigma_2}}{2} \]  

(17)  

the yield surface becomes hyperbolic and convexity is lost. It can be seen that a non-convex yield surface is physically not plausible under plane stress conditions since no yielding will occur under biaxial loading.  

In the invariant plane the convexity condition for the plane stress case is easily seen to lead to the following condition:  

\[ \sigma_s \geq \frac{\sqrt{\sigma_1 \sigma_2}}{2^n} \Rightarrow A_2 \leq \frac{9}{4} \]  

(18)  

Thus, positive values of \( A_2 \) are allowed and the yield curve can actually show a limited positive curvature.  

In Figure 4a, it is shown that convex yield surfaces in principal stress space are guaranteed as long as the yield curve has a finite intersection with the biaxial line:  

\[ \sigma_{im}^2 - A_0 - A_1 p - A_2 p^2 = 0 \]  

\[ \sigma_{im} = \pm \frac{3p}{2} \]  

\[ \frac{9p^2}{4} - A_0 - A_1 p - A_2 p^2 = 0 \]  

(19)  

This last equation will have a real finite solution for the pressure \( p \) iff  

\[ A_1^2 + 4A_0 \left( \frac{9}{4} - A_2 \right) \geq 0 \Rightarrow \sigma_s \geq \frac{\sqrt{\sigma_1 \sigma_2}}{2} \]  

(20)  

Which is equivalent to requiring a convex yield surface in principal stress space. Some examples of convex yield surfaces are shown in Figure 4b. Some remarks are summarized in the table below:
Remarks

\[
\begin{array}{|c|c|}
\hline
\sigma_z / \sqrt{\sigma_t \sigma_c} & A_2 \\
\hline
1 & -18 \quad \nu_p < 0 \\
0.707 & -4.5 \quad \nu_p = 0 \\
0.57735 & 0 \quad \text{von Mises} \quad \nu_p = 0.5 \\
0.5 & 2.25 \quad \text{No biaxial yield} \\
0 & 9 \quad \text{No shear strength} \\
\hline
\end{array}
\]

2.1.2.3 The 3D case

In the full 3D case, the convexity condition is generally more stringent. Again we require the eigenvalues of \( F \) to be non-negative, where \( F \) is now the full 6 by 6 matrix:

\[
\begin{align*}
F_{11} + 2F_{12} & \geq 0 \\
F_{11} - F_{12} & \geq 0 \\
F_{44} & \geq 0
\end{align*}
\]

\[
\Rightarrow \begin{cases} 
3\sigma^2 \geq \sigma_t \sigma_c \\
-\sigma_0 \geq 0
\end{cases}
\]

(21)

Leading to

\[
\sigma_z \geq \sqrt{\sigma_t \sigma_c} > \frac{\sqrt{\sigma_t \sigma_c}}{2}
\]

(22)

Note that the trivial conditions

\[
\begin{align*}
\sigma_z & \geq 0 \\
\sigma_t & \geq 0 \\
\sigma_i & \geq 0 \\
F_0 & \leq 0 
\end{align*}
\]

(23)

are already sufficient to ensure

\[
\begin{align*}
F_{44} & \geq 0 \\
F_{11} & \geq 0 \\
F_{11} - F_{12} = 2F_{44} & \geq 0
\end{align*}
\]

(24)

And consequently, only the first eigenvalue may be negative. Note that the condition that the first eigenvalue must not be negative can be reformulated as

\[
F_{11} + 2F_{12} = -\frac{A_2}{9} \geq 0 \Rightarrow A_2 \leq 0.
\]

(25)

In 3D principal stress space a convex quadratic yield surface corresponds to an ellipsoid, a cylinder or an ellipsoidal paraboloid. A detailed analysis is given in appendix 2.
It should be noted here that convexity in stress space is a sufficient condition for convexity in the invariant space spanned by the pressure and the vonMises stress but the reverse is not the case. Indeed consider the case of a linear relationship between the invariants (Drucker-Prager type law):

![Figure 5: Drucker-Prager yield surface in invariant plane](image)

This is a limiting case for convexity in the invariant plane and

\[
\sqrt{3} \sigma_s = \frac{2\sigma_s \sigma_c}{\sigma_s + \sigma_c} \Rightarrow \sigma_s = \frac{\sqrt{\sigma_s \sigma_c} \cdot 2 \sqrt{\sigma_s \sigma_c}}{\sqrt{3} \ (\sigma_s + \sigma_c)} \leq \frac{\sqrt{\sigma_s \sigma_c}}{\sqrt{3}}
\]  

(26)

Showing that this yield surface is not convex in stress space unless iff:

\[
\sigma_c = \sigma_i
\]  

(27)

This corresponds to the vonMises cylinder. In general, a Drucker-Prager type law is represented by a cone in principal stress space. This surface is strictly spoken not convex because the cone consists of two blades. To complete our discussion a formal derivation of the convexity condition in the invariant plane is given by

\[
f = \sigma_{ym}^2 - A_0 - A_1 p - A_2 p^2
\]

\[
\sigma_{ym} = \sqrt{A_0 - A_1 p + A_2 p^2}
\]

\[
\frac{\partial^2 \sigma_{ym}}{\partial p^2} = \frac{4A_2 (A_0 + A_1 p + A_2 p^2) - (A_1 + 2A_2 p)^2}{2 \sqrt{(A_0 + A_1 p + A_2 p^2)^3}}
\]  

(28)

\[
\frac{\partial^2 \sigma_{ym}}{\partial p^2} \leq 0 \Rightarrow 4A_2 A_0 - A_1^2 \leq 0 \Rightarrow \sigma_s \geq \frac{2\sigma_s \sigma_c}{\sqrt{3} (\sigma_s + \sigma_c)}
\]

2.1.2.4 Alternative formulation

Alternatively a yield surface containing a linear rather than a quadratic term was implemented in SAMP-1.

\[
f = \sigma_{ym} - A_0 - A_1 p - A_2 p^2 \leq 0
\]  

(29)
This formulation does not belong to the class of general quadratic yield surfaces that was just discussed. Coefficients are again easily identified from the uniaxial tension, compression and simple shear tests:

\[
\begin{align*}
\sigma_s \sqrt{3} &= A_0 \\
\sigma_i &= \sigma_s \sqrt{3} - A_1 \frac{\sigma_i}{3} + A_2 \frac{\sigma_i^2}{9} \\
\sigma_c &= \sigma_s \sqrt{3} + A_1 \frac{\sigma_c}{3} + A_2 \frac{\sigma_c^2}{9}
\end{align*}
\]

\[
\Rightarrow A_1 = 3 \left[ \frac{\sigma_i - \sigma_c}{\sigma_i + \sigma_c} - \frac{\sigma_i - \sigma_c}{\sigma_c} \sqrt{3} \frac{\sigma_i - \sigma_c}{\sigma_i \sigma_c} \right] \\
A_2 = 18 \left[ \frac{1}{\sigma_i + \sigma_c} - \frac{\sigma_s \sqrt{3}}{2 \sigma_c} \right]
\]

This formulation may lead to some time savings if a vonMises or Drucker-Prager type yield surface are used (obtained respectively by simply setting $A_1=A_2=0$ and $A_2=0$). No full investigation concerning convexity was performed in this case and in general this formulation resulted in a more difficult fit to test results for different plastics. Convexity in invariant space requires that

\[
A_2 \leq 0 \Rightarrow \sigma_s \sqrt{3} \geq \frac{2 \sigma_i \sigma_c}{\sigma_c + \sigma_i}
\]

### 2.2 Flow rule

Associated flow gives the plastic strain rate in terms of the normal vector to the yield surface:

\[
\dot{\varepsilon}_p = \dot{\lambda} \mathbf{n} = \dot{\lambda} \left\| \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}} \right\| = \dot{\lambda} \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}} \cdot \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}}
\]

We additionally define the volumetric plastic strain rate, deviatoric plastic strain rates and equivalent plastic strain rate in the usual way:

\[
\dot{\varepsilon}_{pv} = tr(\dot{\varepsilon}_p) = \dot{\lambda} \frac{tr \left( \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}} \right)}{\left\| \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}} \right\|}
\]

\[
\dot{\varepsilon}_{pd} = \dot{\varepsilon}_p - \frac{\dot{\varepsilon}_{pv}}{3} \mathbf{d}
\]

\[
\dot{\varepsilon}_p = \frac{2}{3} \dot{\varepsilon}_{pd} : \dot{\varepsilon}_{pd} = \dot{\lambda} \frac{2}{3} \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}} : \frac{\partial \mathbf{f}}{\partial \mathbf{\sigma}}
\]

We have implemented associated flow for the general quadratic yield surface:
\[
\frac{\partial \sigma_{\text{vm}}}{\partial \sigma} = \frac{3}{2\sigma_{\text{vm}}} s \\
\frac{\partial p}{\partial \sigma} = -\frac{1}{3} \delta \Rightarrow \frac{\partial f}{\partial \sigma} = 3s + \frac{A_1 + 2A_2p}{3} \delta
\]

\[
\left\| \frac{\partial f}{\partial \sigma} \right\| = \sqrt{9s : s + \frac{1}{3}(A_1 + 2A_2p)^2} = \sqrt{6\sigma_{\text{vm}}^2 + \frac{1}{3}(A_1 + 2A_2p)^2}
\]

Giving for the volumetric and deviatoric plastic strain rates respectively:

\[
\dot{\varepsilon}_{\text{vp}} = \dot{\lambda}(A_1 + 2A_2p)/\left\| \frac{\partial f}{\partial \sigma} \right\| \\
\dot{\varepsilon}_{\text{dp}} = \dot{\lambda}3s/\left\| \frac{\partial f}{\partial \sigma} \right\|
\]

And the equivalent plastic strain rate follows:

\[
\dot{\varepsilon}_p = \sqrt{\frac{2}{3} \dot{\varepsilon}_{\text{dp}} : \dot{\varepsilon}_{\text{dp}}} = \dot{\lambda} \sqrt{\frac{2}{3} 3s : 3s} / \left\| \frac{\partial f}{\partial \sigma} \right\| \\
\dot{\varepsilon}_p = \dot{\lambda} 2\sigma_{\text{vm}} / \left\| \frac{\partial f}{\partial \sigma} \right\|
\]

It is instructive to derive the ratio of transversal to longitudinal plastic strain rate under uniaxial tensile and compressive loading. This ratio will here be called the plastic Poisson ratio although of course it is by no means a material constant:

\[
\nu_p = \frac{\dot{\varepsilon}_{\text{yp}}}{{\dot{\varepsilon}}_{\text{xp}}} = -\frac{\dot{\varepsilon}_{\text{zp}}}{{\dot{\varepsilon}}_{\text{xp}}}
\]

It will be shown that additional restrictions are placed on the shape of the yield surface in order for the lateral behaviour of the material model to be reasonable under plastic loading. We therefore compute the plastic Poisson ratio in function of the yield surface:

\[
\begin{align*}
\dot{\varepsilon}_{\text{vp}} &= \dot{\lambda}(A_1 + 2A_2p) s_{xx} + \frac{A_1 + 2A_2p}{3} \\
\dot{\varepsilon}_{\text{xp}} &= \dot{\lambda} 3s_{xx} + \frac{A_1 + 2A_2p}{3} \nu_p \\
\nu_p &= \frac{9 + 2A_2 + A_1/3}{18 - 2A_2 - A_1/3} \\
\end{align*}
\]

This last expression gives:

\[
\nu_p = \frac{9 + 2A_2 + A_1/3}{18 - 2A_2 - A_1/3}
\]
Showing that the plastic Poisson ratio is dependent upon the pressure and in particular the lateral behaviour of the material is different in tension and in compression. This can be estimated further as:

\[
\begin{align*}
    p < 0 & \Rightarrow p = -\frac{\sigma_s}{3} \Rightarrow v_p = \frac{9 + 2A_2 - 3A_1}{18 - 2A_2 + 3A_1} \Rightarrow \frac{\sigma_s^2}{\sigma_i + \sigma_c} = \frac{\sigma_c}{\sigma_i + \sigma_c} - 1 \\
    p > 0 & \Rightarrow p = \frac{\sigma_c}{3} \Rightarrow v_p = \frac{9 + 2A_2 + 3A_1}{18 - 2A_2 - 3A_1} \Rightarrow \frac{\sigma_s^2}{\sigma_i + \sigma_c} = \frac{\sigma_s}{\sigma_i + \sigma_c} - 1 
\end{align*}
\]

These equations can be solved for the shear yield:

\[
\begin{align*}
    p < 0 & \Rightarrow v_p = \frac{\sigma_s^2}{\sigma_i + \sigma_c} - 1 \Rightarrow \sigma_s^2 = \frac{\sigma_s}{\sigma_i + \sigma_c} + 1 + v_p \Rightarrow \sigma_s = \sigma_i + \sigma_c \\
    p > 0 & \Rightarrow v_p = \frac{\sigma_s^2}{\sigma_i + \sigma_c} - 1 \Rightarrow \sigma_s^2 = \frac{\sigma_s}{\sigma_i + \sigma_c} + 1 + v_p \Rightarrow \sigma_s = \sigma_i + \sigma_c
\end{align*}
\]

Showing that reasonable values for the plastic Poisson ratio put certain requirements on the yield surface:

\[
\begin{align*}
    0 & \leq v_p \leq 0.5 \Rightarrow \\
    p < 0 & \Rightarrow \sigma_s^2 \leq \frac{2\sigma_s}{3} \leq \frac{\sigma_s^2}{3} \leq \frac{3\sigma_s}{\sigma_i + \sigma_c} \\
    p > 0 & \Rightarrow \sigma_s^2 \leq \frac{2\sigma_s}{3} \leq \frac{\sigma_s^2}{3} \leq \frac{3\sigma_s}{\sigma_i + \sigma_c}
\end{align*}
\]

Whereas convexity required only a lower limit for the shear yield, plausible plastic flow also imposes an upper limit with respect to tensile and compressive yield values. As it will be difficult in general to guarantee a reasonable flow behaviour from three independent measurements in shear, tension and compression, a simplified flow rule has been implemented as the default in SAMP-1. The generally non-associated flow surface is given as:

\[
g = \sigma_{vm}^2 + ap^2
\]

This flow rule is associated iff:

\[
A_1 = 0 \\
A_2 = -\alpha \quad (= \text{cte})
\]

And clearly leads to a constant value for the plastic Poisson ratio:

\[
v_p = \frac{9 - 2\alpha}{18 + 2\alpha} \Rightarrow \alpha = \frac{9 - 2v_p}{2 + v_p}
\]

Plausible flow behaviour just means that:

\[
0 \leq \alpha \leq \frac{9}{2} \Rightarrow 0 \leq v_p \leq 0.5
\]
In SAMP-1 the value of the plastic Poisson coefficient is given by the user, either as a constant or as a load curve in function of the uniaxial plastic strain. This allows to adjust the flow rule of the material to measurements of transversal deformation during uniaxial tensile or compressive testing. This can be important for plastics since often a non-isochoric behaviour is measured.

The possible values for the plastic Poisson ratio and the resulting flow behaviour are illustrated in the sketch below:

\[ \sigma_f = \sigma_{vm} + \alpha p^2 \]

\( \nu_p = 0.5 \quad \alpha = 0 \)

\( \nu_p > 0.5 \quad \alpha < 0 \)

\( \nu_p < 0.5 \quad \alpha > 0 \)

Figure 6: Influence of the flow rule on the plastic Poisson ratio

The volumetric and deviatoric plastic strain rates in this case are given as:

\[ \dot{\varepsilon}_v = \dot{\lambda}(-2\alpha p) / \sqrt{\| \varepsilon_{g} \|} = \frac{\dot{\lambda}(-2\alpha p)}{\sqrt{6\sigma_{vm}^2 + \frac{4}{3}\alpha^2 p^2}} \]

\[ \dot{\varepsilon}_d = \dot{\lambda}3s / \sqrt{\| \varepsilon_{g} \|} = \frac{\dot{\lambda}3s}{\sqrt{6\sigma_{vm}^2 + \frac{4}{3}\alpha^2 p^2}} \]

(48)

In SAMP-1 the formulation is slightly modified and based on a flow rule given as:

\[ g' = \sqrt{\sigma_{vm}^2 + \alpha p^2} \]

(49)

The plastic strain rate computation is not normalized:

\[ \dot{\varepsilon}_p = \dot{\lambda} \frac{\varepsilon_{g}'}{\varepsilon_{\sigma}} \]

(50)

The volumetric and deviatoric plastic strain rates in this case are given as:

\[ \dot{\varepsilon}_v = \dot{\lambda}(-2\alpha p) / 2g' = \frac{\dot{\lambda}(-2\alpha p)}{\sqrt{4\sigma_{vm}^2 + 4\alpha p^2}} \]

\[ \dot{\varepsilon}_d = \dot{\lambda}3s / 2g' = \frac{\dot{\lambda}3s}{\sqrt{4\sigma_{vm}^2 + 4\alpha p^2}} \]

(51)

Which amounts to a different definition of the plastic consistency parameter which of course has to be considered when equivalent plastic strain values are computed.
2.3 Hardening formulation

The hardening formulation is the attractive part of SAMP-1. The formulation is fully tabulated and consequently the user can directly input measurement results from uniaxial tension, uniaxial compression and simple shear tests in terms of load curves giving the yield stress as a function of the corresponding plastic strain. No fitting of coefficients is required. The test results that are reflected in the load curves will be used exactly by SAMP-1 without fitting to any analytical expression. Consequently the hardening will be dependent upon the state of stress and not only upon the plastic strain. The load curves that are expected as input are briefly described here:

\[
\sigma_t = \sigma_i - \frac{\sigma}{E} \left( \varepsilon = \ln \frac{l}{l_0} \right)
\]

Figure 7: Hardening curve in tension and compression

\[
\sigma_c = \sigma_i - \frac{\sigma}{E} \left( \varepsilon = -\ln \frac{l}{l_0} \right)
\]

Figure 8: Hardening curve in shear and tabulated plastic Poisson ratio

The hardening rule now requires us to give the evolution of all hardening parameters as a function of the plastic consistency parameter:

\[
\dot{A}_0 = \frac{\partial A_0}{\partial \lambda} \dot{\lambda}
\]

\[
\dot{A}_1 = \frac{\partial A_1}{\partial \lambda} \dot{\lambda}
\]

\[
\dot{A}_2 = \frac{\partial A_2}{\partial \lambda} \dot{\lambda}
\]

\[
\dot{\alpha} = \frac{\partial \alpha}{\partial \lambda} \dot{\lambda}
\]

The first three equations must be rewritten as follows:

\[
\dot{\varepsilon}_p = \dot{\varepsilon}_i - \frac{\sigma}{E} \left( \varepsilon = \ln \frac{l}{l_0} \right)
\]

\[
\dot{\varepsilon}_c = \dot{\varepsilon}_i - \frac{\sigma}{E} \left( \varepsilon = -\ln \frac{l}{l_0} \right)
\]
\[ A_0 = \frac{3\sigma_s^2}{\sigma_s} \]

\[ \frac{\partial A_0}{\partial \lambda} = \frac{\partial A_0}{\partial \sigma_s} \frac{\partial \sigma_s}{\partial \lambda} + \frac{\partial A_0}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial \lambda} + \frac{\partial A_0}{\partial \sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \]

\[ \frac{\partial A_0}{\partial \sigma_s} = 6\sigma_s \frac{\partial \sigma_s}{\partial \lambda} \]

(53)

\[ A_1 = \frac{9\sigma_t^2}{\sigma_c} \frac{\partial \sigma_t}{\partial \lambda} - \frac{\partial \sigma_t}{\partial \lambda} \]

\[ \frac{\partial A_1}{\partial \lambda} = \frac{\partial A_1}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial \lambda} + \frac{\partial A_1}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial \lambda} + \frac{\partial A_1}{\partial \sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \]

\[ \frac{\partial A_1}{\partial \sigma_t} = 18\sigma_t \frac{\sigma_t - \sigma_i}{\sigma_i \sigma_c} \frac{\partial \sigma_t}{\partial \lambda} - 9 \frac{\sigma_t^2}{\sigma_i \sigma_c} \frac{\partial \sigma_t}{\partial \lambda} + 9 \frac{\sigma_t^2}{\sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \]

(54)

\[ A_2 = 9 \left( \frac{\sigma_c - 3\sigma_s^2}{\sigma_c} \right) \]

\[ \frac{\partial A_2}{\partial \lambda} = \frac{\partial A_2}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial \lambda} + \frac{\partial A_2}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial \lambda} + \frac{\partial A_2}{\partial \sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \]

\[ \frac{\partial A_2}{\partial \sigma_t} = 9 \left( \frac{6\sigma_t}{\sigma_i \sigma_c} \frac{\partial \sigma_t}{\partial \lambda} + \frac{3\sigma_s^2}{\sigma_i \sigma_c} \frac{\partial \sigma_t}{\partial \lambda} + \frac{3\sigma_t^2}{\sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \right) \]

(55)

If we add the following conversion:

\[ \frac{\partial \sigma_s}{\partial \lambda} = \frac{\partial \sigma_s}{\partial \varepsilon_{ps}} \frac{\partial \varepsilon_{ps}}{\partial \lambda} \]

\[ \frac{\partial \sigma_t}{\partial \lambda} = \frac{\partial \sigma_t}{\partial \varepsilon_{pt}} \frac{\partial \varepsilon_{pt}}{\partial \lambda} \]

\[ \frac{\partial \sigma_c}{\partial \lambda} = \frac{\partial \sigma_c}{\partial \varepsilon_{pc}} \frac{\partial \varepsilon_{pc}}{\partial \lambda} \]

(56)

Then the hardening mechanism is fully determined by performing three table lookups during every iteration of every timestep. The table lookups give the yield stress and the tangent as a function of plastic strain for every experiment:

\[ \varepsilon_{ps} \Rightarrow \sigma_s, \frac{\partial \sigma_s}{\partial \varepsilon_{ps}} \]

\[ \varepsilon_{pt} \Rightarrow \sigma_t, \frac{\partial \sigma_t}{\partial \varepsilon_{pt}} \]

\[ \varepsilon_{pc} \Rightarrow \sigma_c, \frac{\partial \sigma_c}{\partial \varepsilon_{pc}} \]

(57)

What remains to be done is to establish the relationship between the plastic consistency parameter and the plastic strains that were measured under uniaxial tension/compression and simple shear. To achieve this we start by computing the equivalent plastic strain rate as a function of the plastic consistency parameter. Note that the hardening rule must be carefully considered at this point:
\[
\dot{\varepsilon}_p = \frac{2}{\sqrt{3}} \dot{\varepsilon}_\text{dp} : \dot{\varepsilon}_\text{dp}
\]

\[
\begin{align*}
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{2g} \sqrt{\frac{2}{3} \frac{\partial \sigma_{\text{vm}}}{\partial \varepsilon} : \frac{\partial \sigma_{\text{vm}}}{\partial \sigma}} \\
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{2g} \sqrt{\frac{2}{3} 3s : 3s} \\
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{g} \sigma_{\text{vm}} \\
\end{align*}
\]

\[
\text{default} \quad \text{associated}
\]

\[
\begin{align*}
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{2g} \left[ \frac{2}{3} \frac{\partial \sigma_{\text{vm}}}{\partial \varepsilon} : \frac{\partial \sigma_{\text{vm}}}{\partial \sigma} \right] \\
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{2g} \left[ \frac{2}{3} 3s : 3s \right] \\
\dot{\varepsilon}_p &= \frac{\dot{\lambda}}{g} 2\sigma_{\text{vm}} \\
\end{align*}
\]

And then establish the relationship between the individual plastic strain rate values and the consistency parameter, for the uniaxial case we get

\[
\begin{align*}
\dot{\varepsilon}_p &= \left( \begin{array}{ccc}
\dot{\varepsilon}_{\text{pct}} & 0 & 0 \\
0 & -\nu_p \dot{\varepsilon}_{\text{pct}} & 0 \\
0 & 0 & -\nu_p \dot{\varepsilon}_{\text{pct}} \\
\end{array} \right) \\
\dot{\varepsilon}_p &= \sqrt{2 \dot{\varepsilon}_\text{dp} : \dot{\varepsilon}_\text{dp}} = |\dot{\varepsilon}_{\text{pct}}| \frac{2}{3} \left( 1 + \nu_p \right) \\
\end{align*}
\]

\[
|\dot{\varepsilon}_{\text{pct}}| = \dot{\varepsilon}_p \frac{3}{2(1 + \nu_p)} = \left\{ \begin{array}{c}
\frac{\dot{\lambda}}{g} \sigma_{\text{vm}} \frac{3}{2} \\
\frac{2\sigma_{\text{vm}}}{|n|} \frac{3}{2} \\
\end{array} \right. \\
\]

And similarly we obtain for the plastic strain rate under shear loading:

\[
\dot{\varepsilon}_p = \dot{\varepsilon}_\text{dp} = \left( \begin{array}{ccc}
0 & \dot{\varepsilon}_{\text{ps}} & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{array} \right) \\
\dot{\varepsilon}_p = \sqrt{\frac{2}{3} \dot{\varepsilon}_\text{dp} : \dot{\varepsilon}_\text{dp}} = \frac{2}{\sqrt{3}} |\dot{\varepsilon}_{\text{ps}}| \\
\]

\[
|\dot{\varepsilon}_{\text{ps}}| = \dot{\varepsilon}_p \frac{\sqrt{3}}{2} = \left\{ \begin{array}{c}
\frac{\dot{\lambda}}{g} \sigma_{\text{vm}} \frac{\sqrt{3}}{2} \\
\frac{\dot{\lambda}}{2\sigma_{\text{vm}}} \frac{\sqrt{3}}{2} \\
\end{array} \right. \\
\]

These equations always allow to determine the individual plastic strain values and thus the abscissa values for the table-lookup operations from the plastic consistency parameter.
\[ |\dot{\varepsilon}_{pcr}| = \dot{\varepsilon}_p \frac{3}{2(1 + \nu_p)} \]
\[ = \left\{ \frac{\dot{\lambda}}{g} \frac{\sigma_{vm}}{2(1 + \nu_p)} \right\} \frac{3}{2(1 + \nu_p)} \Rightarrow |\varepsilon_{pcr}| = \left[ \frac{\sigma_{vm}}{2(1 + \nu_p)} \right] d\lambda \]
\[ (61) \]
\[ |\dot{\varepsilon}_p| = \dot{\varepsilon}_p \frac{\sqrt{3}}{2} = \left\{ \frac{\dot{\lambda}}{g} \frac{\sigma_{vm}}{2(1 + \nu_p)} \right\} \frac{\sqrt{3}}{2} \Rightarrow |\varepsilon_{p}| = \left[ \frac{\sigma_{vm}}{2(1 + \nu_p)} \right] d\lambda \]
\[ (62) \]
For the case of non-associated flow and a constant plastic Poisson ratio the integration is easily done analytically:
\[ |\dot{\varepsilon}_{pcr}| = \dot{\lambda} \frac{\sigma_{vm}}{g} \frac{3}{2(1 + \nu_p)} \Rightarrow |\varepsilon_{pcr}| = \dot{\lambda} \frac{3}{2} \]
\[ (63) \]
and
\[ |\dot{\varepsilon}_p| = \dot{\lambda} \frac{\sqrt{3}}{2} \Rightarrow |\varepsilon_p| = \dot{\lambda} \frac{\sqrt{3}}{2} \]
\[ (64) \]
In this case the conversion of the tangent values that result from the table lookups is equally trivial:
\[ \frac{\partial \sigma_s}{\partial \lambda} = \frac{\sqrt{3}}{2} \frac{\partial \sigma_s}{\partial \varepsilon_p} \]
\[ \frac{\partial \sigma_t}{\partial \lambda} = \frac{\sqrt{3}}{2(1 + \nu_p)} \frac{\partial \sigma_t}{\partial \varepsilon_p} \]
\[ \frac{\partial \sigma_c}{\partial \lambda} = \frac{\sqrt{3}}{2(1 + \nu_p)} \frac{\partial \sigma_c}{\partial \varepsilon_p} \]
\[ (65) \]
In the case of associated flow the conversion factors are not constants and must be evaluated at each timestep:
\[ \frac{\partial \sigma_s}{\partial \lambda} = \frac{\partial \sigma_s}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial \lambda} \approx \frac{\partial \sigma_s}{\partial \varepsilon_p} \frac{2\sigma_{vm}}{\sigma_{vm}} \frac{\sqrt{3}}{2} \]
\[ \frac{\partial \sigma_t}{\partial \lambda} = \frac{\partial \sigma_t}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial \lambda} \approx \frac{\partial \sigma_t}{\partial \varepsilon_p} \frac{2\sigma_{vm}}{\sigma_{vm}} \frac{3}{2(1 + \nu_p)} \]
\[ \frac{\partial \sigma_c}{\partial \lambda} = \frac{\partial \sigma_c}{\partial \varepsilon_p} \frac{\partial \varepsilon_p}{\partial \lambda} \approx \frac{\partial \sigma_c}{\partial \varepsilon_p} \frac{2\sigma_{vm}}{\sigma_{vm}} \frac{3}{2(1 + \nu_p)} \]
\[ (66) \]
2.4 Rate effects

Plastics are usually highly rate dependent. A proper viscoplastic consideration of the rate effects is therefore important in the numerical treatment of the material law. Data to determine the rate dependency are based on uniaxial dynamic testing. If dynamic tests are available, then the load curve defining the yield stress in uniaxial tension is simply replaced by a table definition containing multiple load curves corresponding to different values of the plastic strain rate. This is illustrated in the figure below.

![Figure 9: Tensile hardening curve from dynamic tensile tests](image)

The table lookups then involve determining static yield values for tension, compression and shear, as well as dynamic yield values in tension. Tangents with respect to the strain rate must also be evaluated:

\[
\dot{\varepsilon}_p = \frac{\sigma_t}{E} \quad (\varepsilon = \ln \frac{l}{l_0})
\]

It is then simply assumed that the rate effect in compression and shear is similar to the rate effect under tensile loading:

\[
\sigma_s = \sigma_{s0} \frac{\sigma_t}{\sigma_{t0}} \quad \sigma_c = \sigma_{c0} \frac{\sigma_t}{\sigma_{t0}}
\]

Tangents to the yield surface are then computed consistently with the previous assumption:
where

$$\text{afac} = \left. \frac{\partial \sigma_y}{\partial \varepsilon} \right|_{0}^{-1}$$

(70)

Although this approach is certainly questionable since rate effects may depend upon the state of stress, it is justified by the fact that dynamic test results are not easily obtainable under shear and compression. A generalisation involving table-type definitions for all three types of experiments could be implemented relatively fast if needed.

### 2.5 Damage and failure

Numerous damage models can be found in the literature. Probably the simplest concept is elastic damage where the damage parameter (usually written as d) is a function of the elastic energy and effectively reduces the elastic modulae of the material. In the case of ductile damage, d is a function of plastic straining and effects the yield stress rather then the elastic modulae. This is equivalent to plastic softening. In more sophisticated damage models, d depends on both the plastic straining and the elastic energy (and maybe other factors) and effects yield stress as well as elastic modulae. (see /Lemaitre/).

A simple damage model was added to the SAMP-1 material law where the damage parameter d is a function of plastic strain only. A load curve must be provided by the user giving d as a function of the (true) plastic strain under uniaxial tension. The value of the critical damage Dc leading to rupture is then the only other required additional input. The implemented damage model is isotropic.

The implemented model then uses the notion of effective cross section, which is the true cross section of the material minus the cracks that have developed. We will use the following notation:

$$A_0 \quad \Rightarrow \quad \text{undeformed cross section}$$
$$A \quad \Rightarrow \quad \text{deformed or current cross section}$$
$$A_{eff} = A(1-d) \quad \Rightarrow \quad \text{effective cross section}$$
We define the effective stress as the force divided by the effective cross section:

\[
\sigma = \frac{f}{A}
\]

\[
\sigma_{\text{eff}} = \frac{f}{A_{\text{eff}}} = \frac{f}{A(1-d)} = \frac{\sigma}{1-d}
\]

(71)

Which allows to define an effective yield stress:

\[
\sigma_{y,\text{eff}} = \frac{\sigma_y}{1-d}
\]

(72)

And simply apply the principle of strain equivalence stating that using the undamaged modulus, the effective stress corresponds to the same elastic strain as the true stress using the damaged modulus:

\[
E = \frac{\sigma_{\text{eff}}}{\varepsilon_e}
\]

\[
E_d = \frac{\sigma}{\varepsilon_e} = E(1-d)
\]

(73)

Note that the plastic strains are then also the same:

\[
\varepsilon_p = \varepsilon - \frac{\sigma_{\text{eff}}}{E} = \varepsilon - \frac{\sigma}{E_d}
\]

(74)

No damage will occur under pure elastic deformation with this model. The case of a material that is perfectly plastic in it’s undamaged state is illustrated by the figure below:

*Figure 10: Damage parameter from uniaxial tensile test*
It can be seen that the damage parameter effectively reduces the elastic modulus. Consequently if unloading is performed at different strain values during the uniaxial tensile test, the different unloading slopes allow to estimate the damage parameter for a given plastic strain:

\[
d(\varepsilon_{pt}) = 1 - \frac{E_d(\varepsilon_{pt})}{E}
\]

(75)

The damage model will thus be used essentially to fit the unloading behaviour of the material. The two stage process of determining input data from a measured true stress/strain curve is illustrated below. In a first step the damage curve is derived:

\[
\sigma = \sigma_{eff}(1 - d)
\]

\[
E_d = E(1 - d)
\]

\[
E_{d E} = E_i(1 - d)
\]

Figure 11: Damage

And in a second step the hardening curve is determined in terms of effective stresses:

\[
\sigma_{y eff} = \frac{\sigma}{1 - d}
\]

\[
\sigma_{yi eff} = \frac{\sigma}{E_{yi}}
\]

\[
E_{yi eff} = E_i(1 - d)
\]

Figure 12: Determination of damage as a function of plastic strain

Figure 13: Conversion from true stress to effective hardening curve
As usual the failure strain corresponds to the point where \( d = 0 \) and the rupture strain corresponds to the point where \( d \) reaches the critical value \( D_c \).

If the damage load curve is given a negative identification number, then the hardening curves can be input in terms of true stresses and the input preparation is performed as if there were no damage:

![Figure 14: Conversion from true stress to hardening curve](image)

In this case the numerically computed stress values will correspond to the input data and the damage model will seem to affect only the elastic modulae and thus the unloading/reloading behaviour of the material.

### 2.6 Crazing

Many plastics and PP-EPDM in particular show a localized deformation process called crazing. The material will typically change colour and turn white in the craze. From a mechanical point of view crazing can be identified with a permanent increase of volume (volumetric plastic straining) and a low biaxial strength.

To simulate crazing it may therefore be desirable to consider biaxial test data in the numerical model. In SAMP-1 the hardening curve resulting from a biaxial tensile test is therefore optional as input. The curve should give yield stress as a function of volumetric strain and care must be taken to use the correct biaxial modulus when transforming from true strain to plastic strain. This is illustrated in the figure below.

![Figure 15: Biaxial hardening curve](image)

For the table lookup operations the relationship must be established between the plastic consistency parameter and the biaxial plastic strain as defined above. This is easily done for the case of non-associated plasticity with a constant plastic Poisson ratio. We start establishing the equation for the biaxial plastic strain rate.
\[ \dot{\varepsilon}_p = \begin{pmatrix} \dot{\varepsilon}_{pb} & 0 & 0 \\ 0 & \dot{\varepsilon}_{pb} & 0 \\ 0 & 0 & -\frac{2v_p}{1-v_p} \dot{\varepsilon}_{pb} \end{pmatrix} \]

\[ \dot{\varepsilon}_p = \sqrt{\frac{2}{3}} \dot{\varepsilon}_p : \dot{\varepsilon}_p = \sqrt{\frac{2}{3}} \dot{\varepsilon}_{pb} \frac{2(1 + v_p)}{3(1 - v_p)} \]

\[ |\dot{\varepsilon}_{pb}| = \dot{\varepsilon}_p \frac{3(1 - v_p)}{2(1 + v_p)} = \lambda \frac{\sigma_{\text{vm}}}{g} \frac{3(1 - v_p)}{2(1 + v_p)} \]

And as in the case of uniaxial and shear loading this can be integrated explicitly:

\[ |\dot{\varepsilon}_{pb}| = \dot{\varepsilon}_p \frac{3(1 - v_p)}{2(1 + v_p)} = \lambda \frac{3(1 - v_p)}{4(1 + v_p)} \]

Allowing to perform a table-lookup yielding the biaxial strength as a function of the plastic consistency parameter, analogously to 2.3. If biaxial data are provided in addition to uniaxial tension, compression and shear, then the quadratic yield surface no longer allows perfect fitting of all data. A best possible solution is then implemented using a least squares approach. The yield functions are written for all 4 load cases

\[ f_i = \sigma_i^2 - A_0 + A_1 \frac{\sigma_i}{3} - A_2 \frac{\sigma_i^2}{9} \]

\[ f_c = \sigma_c^2 - A_0 + A_1 \frac{\sigma_c}{3} - A_2 \frac{\sigma_c^2}{9} \]

\[ f_s = 3\sigma_s^2 - A_0 \]

\[ f_b = \sigma_b^2 - A_0 + 2A_1 \frac{\sigma_b}{3} - 4A_2 \frac{\sigma_b^2}{9} \]

And coefficients are determined in the classical way:

\[ f = f_i^2 + f_c^2 + f_s^2 + f_b^2 \]

\[ \frac{\partial f}{\partial A_0} = 0 \Rightarrow f_i + f_c + f_s + f_b = 0 \]

\[ \frac{\partial f}{\partial A_1} = 0 \Rightarrow f_i \sigma_i - f_c \sigma_c + 2f_s \sigma_s = 0 \]

\[ \frac{\partial f}{\partial A_2} = 0 \Rightarrow f_i \sigma_i^2 + f_c \sigma_c^2 + 4f_s \sigma_s^2 = 0 \]

The following three equations must then be solved each timestep:
\[
\begin{align*}
\left(\sigma_i^2 + \sigma_i^2 + 3\sigma_i^2 + \sigma_i^2\right) - 4A_0 + \left(\sigma_0 - \sigma_0 + 2\sigma_0\right)A_1 - \left(\sigma_0^2 + \sigma_0^2 + 4\sigma_0^2\right)A_2 &= 0 \\
\left(\sigma_i^4 + \sigma_i^4 + 4\sigma_i^4\right) - A_0\left(\sigma_i^2 + \sigma_i^2 + 4\sigma_i^2\right) + \left(\sigma_0^2 - \sigma_0^2 + 8\sigma_0^2\right)A_1 - \left(\sigma_0^4 + \sigma_0^4 + 16\sigma_0^4\right)A_2 &= 0 \\
\left(\sigma_i^3 - \sigma_i^3 + 2\sigma_i^3\right) - A_0\left(\sigma_i - \sigma_i + 2\sigma_i\right) + \left(\sigma_0^2 - \sigma_0^2 + 4\sigma_0^2\right)A_1 - \left(\sigma_0^3 - \sigma_0^3 + 8\sigma_0^3\right)A_2 &= 0 
\end{align*}
\]

(80)

However we also need to determine the tangents to the expanding yield surface. We write the equations above in matrix form and take the derivative:

\[
A(\sigma)\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = b(\sigma) \quad \Rightarrow \quad A^{-1}(\sigma_i)b(\sigma_i)
\]

(81)

\[
\frac{\partial A(\sigma)}{\partial \sigma_i}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} + A(\sigma)\frac{\partial}{\partial \sigma_i}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \frac{\partial b(\sigma_i)}{\partial \sigma_i}
\]

The last equation is now solved for the 12 tangent values:

\[
\frac{\partial}{\partial \sigma_i}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = A^{-1}(\sigma_i)\left[ \frac{\partial b(\sigma_i)}{\partial \sigma_i} - \frac{\partial A(\sigma_i)}{\partial \sigma_i}\begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} \right] \quad = \begin{pmatrix} \sigma_i \\ \sigma_i \\ \sigma_i \end{pmatrix}
\]

(82)

In many practical cases where biaxial data are available, some other test result will be missing. In these cases the missing data is generated internally from the biaxial test result assuming a quadratic isotropic yield surface and the material law formulation is not changed. Details concerning these data conversions can be found in appendix 1.

2.7 Numerical implementation

2.7.1.1 The rate-independent plasticity iterations

To solve the equations of elasto-plasticity, the explicit cutting plane algorithm was selected. A complete review of the implementation is given below and is based on /Hughes, Simo&Hughes/. Consider the set of equations defining the SAMP-1 material law:

\[
\sigma = 2\bar{\sigma}\left(\varepsilon - \varepsilon^p\right) + K\left(\varepsilon - \varepsilon^p\right) \quad \text{material law}
\]

(83)

\[
\dot{q} = \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} = \dot{\varepsilon}h = \lambda \begin{pmatrix} \frac{\partial A_0}{\partial \lambda} \\ \frac{\partial A_1}{\partial \lambda} \\ \frac{\partial A_2}{\partial \lambda} \end{pmatrix} \quad \text{hardening rule}
\]

\[
\dot{\varepsilon}^p = \dot{\lambda}r = \lambda \frac{\partial \varepsilon^p}{\partial \sigma} \quad \text{flow rule}
\]

\[
f = \begin{cases} 
\sigma^\text{in} - A_0 - A_1p - A_2p^2 \leq 0 & \text{iquad} = 1 \\
\sigma^\text{in} - A_0 - A_1p - A_2p^2 \leq 0 & \text{iquad} = 0 
\end{cases}
\]

\[
\text{yield surface}
\]
Here we can further specify the flow rule as follows:

\[
\begin{align*}
\text{iquad} = 0/1 & \quad \text{if } \alpha \geq 0 \\
\Rightarrow \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma} = \dot{\lambda} \frac{9s - 2ap\delta}{6\sigma^2 + ap^2} \\
\text{iquad} = 1 & \quad \text{if } \alpha > 0 \\
\Rightarrow \dot{\varepsilon}^p = \dot{\lambda} \frac{\partial g}{\partial \sigma} = \dot{\lambda} \frac{9s + (A_1 + A_2 p)\delta}{\sqrt{6\sigma^2 + (A_1 + A_2 p)^2}}
\end{align*}
\]

It should be noted that no associated flow has been implemented for the alternative yield condition (iquad=0).

In step 1 of the plasticity algorithm the iteration counter \( k \) is set to zero and plastic train rates are initialised to zero, the hardening parameters are set to their values at the previous timestep:

\[
k = 0 \\
\varepsilon_n^{p(0)} = 0 \\
\begin{pmatrix}
A_{0,n+1}^{(0)} \\
A_{1,n+1}^{(0)} \\
A_{2,n+1}^{(0)} \\
\alpha_n^{(0)}
\end{pmatrix} = \\
\begin{pmatrix}
A_{0,n} \\
A_{1,n} \\
A_{2,n} \\
\alpha_n
\end{pmatrix}
\]

In step 2a, we compute new stresses (elastic trial stress at the first iteration)

\[
\sigma_n^{(k)} = \sigma_n + c(\varepsilon_{n+1} - \varepsilon_n^{p(k)}) \Delta t.
\]  

In step 2b, we evaluate the yield function, if the computed stress is below the yield surface, we are finished (note that this step involves computing the yield function coefficients from the tabulated data)

\[
f_n^{(k)} = \sigma_{\text{vm},n+1}^{(k)} - A_{0,n+1}^{(k)} - A_{1,n+1}^{(k)}p_{n+1}^{(k)} - A_{2,n+1}^{(k)}P_{n+1}^{(k)}.
\]

It the state of stress is below or on the yield surface, convergence has been obtained:

\[
\left| f_n^{(k)} \right| \leq \text{tol} \Rightarrow \begin{cases} 
\varepsilon_n^{p} = \varepsilon_n^{p(k)} \\
A_{n+1} = A_{n+1}^{(k)}
\end{cases}
\]

If the yield condition is not fulfilled we need to go to step 3 and compute the increment of the plastic consistency parameter, as a first step this involves computing the hardening functions from the tabulated input as described under paragraph 2.3.

\[
h_{n+1}^{(k)} = \begin{pmatrix}
\frac{\partial A_0}{\partial \lambda} \\
\frac{\partial A_1}{\partial \lambda} \\
\frac{\partial A_2}{\partial \lambda} \\
\frac{\partial \alpha}{\partial \lambda}
\end{pmatrix}_{n+1}^{(k)} = \text{tabulated}(\varepsilon_{n+1}^{p(k)})
\]
Similarly we compute the flow direction and the derivatives of the yield function based on the current trial state:

\[
\left( \frac{\partial f}{\partial \sigma} \right)_{n+1}^{(k)} = 3s_{n+1}^{(k)} + \frac{A_{1,n+1}^{(k)} + 2A_{2,n+1}^{(k)}P_{n+1}^{(k)}}{3} \delta \\
\mathbf{r}_{n+1}^{(k)} = \frac{9s_{n+1}^{(k)} - 2\alpha_{n+1}^{(k)}P_{n+1}^{(k)}\delta}{6\sigma_{min,n+1}^{(k)} + \alpha_{n+1}^{(k)}P_{n+1}^{(k)}}
\]

(89)

Then the consistency condition is enforced at time n+1 and linearized around the current state, this yields for the increment of the plastic consistency parameter:

\[
\Delta \lambda_{n+1}^{(k)} = \frac{f_{n+1}^{(k)}}{\left( \frac{\partial f}{\partial \sigma} \right)_{n+1}^{(k)} : \mathbf{c}_{n+1}^{(k)} \mathbf{r}_{n+1}^{(k)} - \left( \frac{\partial f}{\partial \mathbf{q}} \right)_{n+1}^{(k)} : \mathbf{h}_{n+1}^{(k)}}
\]

(90)

Becomes:

\[
\Delta \lambda_{n+1}^{(k)} = \frac{f_{n+1}^{(k)}}{3s_{n+1}^{(k)} + \frac{A_{1,n+1}^{(k)} + 2A_{2,n+1}^{(k)}P_{n+1}^{(k)}}{3} \delta : \mathbf{c}_{n+1}^{(k)} \left( \frac{9s_{n+1}^{(k)} - 2\alpha_{n+1}^{(k)}P_{n+1}^{(k)}\delta}{6\sigma_{min,n+1}^{(k)} + \alpha_{n+1}^{(k)}P_{n+1}^{(k)}} \right) - \left( -p_{n+1}^{(k)} - p_{n+1}^{(k)} \right) \left( \begin{array}{c} \frac{\partial A_0}{\partial \lambda} \\ \frac{\partial A_1}{\partial \lambda} \\ \frac{\partial A_2}{\partial \lambda} \\ \frac{\partial \alpha}{\partial \lambda} \end{array} \right)_{n+1}}
\]

Finally in step 4 we update plastic strains and stresses:

\[
\mathbf{s}_{n+1}^{(k+1)} = \mathbf{s}_{n+1}^{(k)} - \mathbf{c} \Delta \lambda_{n+1}^{(k)} \mathbf{p}_{n+1}^{(k)}
\]

(91)
Due to convergence problems in the case of non convex yield surfaces, an option was implemented to guarantee a convex shape of the yield surface:

\[ i_{\text{convex}} = 1 \quad \Rightarrow \quad \sigma_s = \max \left( \sigma_{s1}, \sqrt{\frac{\sigma_s \sigma_s}{3}} \right) \]  

The alternative yield surface

\[ f = \sigma_{vm} - A_0 - A_1 p - A_2 p^2 \]  

offers algorithmic advantages in the isochoric case. Indeed we can write the consistency condition for both formulations as

\[ \sigma_{vm,r+1}^2 - A_{2,r+1} p_{r+1}^2 - A_{1,r+1} p_{r+1} - A_{0,r+1} = 0 \quad i_{\text{quad}} = 1 \]

\[ \sigma_{vm,r+1}^2 - A_{2,r+1} p_{r+1}^2 - A_{1,r+1} p_{r+1} - A_{0,r+1} = 0 \quad i_{\text{quad}} = 0 \]  

Or at the first iteration using the elastic trial stresses (indexed e)

\[ \sigma_{vm,e,r+1} \left( 1 - \Delta \lambda \frac{3G}{g_{e,r+1}} \right) - \left( A_{2,e} + \frac{\partial A_{2}}{\partial \lambda} \Delta \lambda \right) p_{e,r+1}^2 \left( 1 - \Delta \lambda \frac{\alpha K}{g_{e,r+1}} \right)^2 - \]

\[ \left( A_{4,e} + \frac{\partial A_{4}}{\partial \lambda} \Delta \lambda \right) p_{e,r+1} \left( 1 - \Delta \lambda \frac{\alpha K}{g_{e,r+1}} \right) - \left( A_{6,e} + \frac{\partial A_{6}}{\partial \lambda} \Delta \lambda \right) = 0 \]  

\[ \sigma_{vm,e,r+1}^2 \left( 1 - \Delta \lambda \frac{3G}{g_{e,r+1}} \right)^2 - \left( A_{2,e} + \frac{\partial A_{2}}{\partial \lambda} \Delta \lambda \right) p_{e,r+1}^2 \left( 1 - \Delta \lambda \frac{\alpha K}{g_{e,r+1}} \right)^2 - \]

\[ \left( A_{4,e} + \frac{\partial A_{4}}{\partial \lambda} \Delta \lambda \right) p_{e,r+1} \left( 1 - \Delta \lambda \frac{\alpha K}{g_{e,r+1}} \right) - \left( A_{6,e} + \frac{\partial A_{6}}{\partial \lambda} \Delta \lambda \right) = 0 \]  

These equations are third order in the increment of the plastic consistency parameter. Linearization around the elastic trial stress gives
\[ \Delta \lambda = \frac{\frac{3G}{g_{e,\alpha+1}} - \sigma_{\text{iso},e,\alpha+1} \Delta \lambda}{3G} \left( A_{2,\alpha} + \frac{\partial A_4}{\partial \Delta \lambda} \right) p_{e,\alpha+1}^2 - \left( A_{4,\alpha} + \frac{\partial A_6}{\partial \Delta \lambda} \right) = 0 \]

\[ \Delta \lambda = \frac{\frac{3G}{g_{e,\alpha+1}} - \sigma_{\text{iso},e,\alpha+1} \Delta \lambda}{3G} \left( A_{2,\alpha} + \frac{\partial A_4}{\partial \Delta \lambda} \right) p_{e,\alpha+1}^2 - \left( A_{4,\alpha} + \frac{\partial A_6}{\partial \Delta \lambda} \right) = 0 \]

However in the isochoric case we have \( \alpha = 1 \) and the full equations reduce to:

\[ \sigma_{\text{iso},e,\alpha+1} \left( 1 - \Delta \lambda \right) - \left( A_{2,\alpha} + \frac{\partial A_4}{\partial \Delta \lambda} \right) p_{e,\alpha+1}^2 - \left( A_{4,\alpha} + \frac{\partial A_6}{\partial \Delta \lambda} \right) = 0 \]

\[ \sigma_{\text{iso},e,\alpha+1} \left( 1 - \Delta \lambda \right) - \left( A_{2,\alpha} + \frac{\partial A_4}{\partial \Delta \lambda} \right) p_{e,\alpha+1}^2 - \left( A_{4,\alpha} + \frac{\partial A_6}{\partial \Delta \lambda} \right) = 0 \]

Which is linear in the increment of the consistency parameter for \( \text{quad} = 0 \). Consequently in this case convergence is guaranteed in a single iteration (radial return).

To complete this overview we add a few considerations about the stress projection in the SAMP-1 with non-associated flow and constant plastic Poisson ratio. In this case the stress projection during one cutting plane iteration can be written as follows:

\[ p_{\alpha+1}^{k+1} - p_{\alpha+1}^{k} = K_{p,\alpha+1}^{k+1} \Rightarrow p_{\alpha+1}^{k+1} = p_{\alpha+1}^{k} \left( 1 - \frac{\alpha K}{3G} \frac{\Delta \lambda}{g_{\alpha+1}^k} \right) \]

\[ s_{\alpha+1}^{k+1} - s_{\alpha+1}^{k} = 2G K_{p,\alpha+1}^{k+1} \Rightarrow s_{\alpha+1}^{k+1} = s_{\alpha+1}^{k} \left( 1 - \frac{3G \Delta \lambda}{g_{\alpha+1}^k} \right) \]

\[ \frac{\alpha K}{3G} = \frac{\left( 1 + \nu \right) \left( 1 - 2\nu \right)}{\left( 1 - 2\nu \right) \left( 1 + \nu \right)} \]

From the above it is clear that in general the stress return is not parallel to the plastic strain rate and thus not orthogonal to the flow surface.

The stress return can be visualized in the invariant plane as in the figure below:
Clearly the stress projection is radial in stress space iff:

\[ v = v_p \Rightarrow \frac{\alpha K}{3G} = \frac{(1 + \nu)(1 - 2\nu_p)}{(1 - 2\nu)(1 + \nu_p)} = 1 \]  

(99)

Similarly the stress projection in the invariant plane is always vertical, thus radial in deviatoric space and orthogonal to the flow surface iff:

\[ v_p = 0.5 \Rightarrow \frac{\alpha K}{3G} = \frac{(1 + \nu)(1 - 2\nu_p)}{(1 - 2\nu)(1 + \nu_p)} = 0 \]  

(100)

The stress return will be parallel to the plastic strain rate and orthogonal to the flow surface iff:

\[ \sigma_{n+1}^{k+1} - \sigma_n^k = E_{p,n+1}^{k} \sigma_n = \sigma_n \]  

(101)

\[ v = 0 \Rightarrow \frac{\alpha K}{3G} = \frac{2(1 - 2\nu_p)}{9(1 + \nu_p)} \]

In order to avoid certain convergence problems we have optionally:

\[ ic_{convex} = 2 \Rightarrow v = \min(v, v_p) \]  

(102)

2.7.1.2 The plane stress iterations

In the case of shell elements a plane state of stress must be obtained. Thus in the element local system we must have that:

\[ \sigma_{zz} = 0 \]  

(103)

This is done by the usual secant iteration as described in /LSDYNA theoretical manual/. In the following equations the superscript indicates the iteration number. The first two estimates correspond to a fully elastic and a fully plastic strain increment respectively.
\[
\Delta \varepsilon_{33}^1 = -\frac{\sigma_{33} + \lambda \left( \Delta \varepsilon_{11} + \Delta \varepsilon_{22} \right)}{\lambda + 2 \mu} \Rightarrow \sigma_{33}^1 \\
\Delta \varepsilon_{33}^2 = -\frac{\nu}{1 - \nu} \left( \Delta \varepsilon_{11} + \Delta \varepsilon_{22} \right) \Rightarrow \sigma_{33}^2 \\
n \geq 3
\]

\[
\Delta \varepsilon_{33}^{n+1} = \Delta \varepsilon_{33}^{n-1} - \frac{\Delta \varepsilon_{33}^{n} - \Delta \varepsilon_{33}^{-n-1}}{\sigma_{33}^{n-1} - \sigma_{33}^{n-1}} \Rightarrow \sigma_{33}^{n+1} \\
\left| \frac{\Delta \varepsilon_{33}^{n} - \Delta \varepsilon_{33}^{n-1}}{\Delta \varepsilon_{33}^{n-1}} \right| < 10^{-4}
\]

In the previous we have in the usual notations :

\[
\mu = G \\
\lambda = K - \frac{2}{3} G \\
\frac{\lambda}{\lambda + 2 \mu} = \frac{\nu}{1 - \nu} = \frac{K + 4G/3}{K - 2G/3}
\]

The maximum number of iterations is 3 by default and can be reset by the user. The plane stress iteration is an outer loop that allows to generate new starting values for the out-of-plane strain in each plasticity iteration. Due to convergence problems in the case of associated plasticity, we imposed

\[
\nu_p \leq 0.5
\]

at this level.

2.7.1.3 The damage model

The implementation of the damage model is trivially simple since the entire plasticity algorithm is performed in terms of effective stresses without any modification.

\[
d_n = f(\varepsilon_{pt,n}) \\
\sigma_{eff,n} = \frac{\sigma_n}{1 - d_n} \\
\ldots \\
d_{n+1} = f(\varepsilon_{pt,n+1}) \\
\sigma_{n+1} = \sigma_{eff,n+1}(1 - d_{n+1})
\]

In case the damage load curve has a negative identification number, the algorithm just amounts to modifying the elastic modulus during each iteration :

\[
E_{d,n}^k = E(1 - d_n^k(\varepsilon_{pt,n}^k))
\]

2.7.1.4 Rate effects

A constitutive law for visco-plastic materials is given a.o. in /Hughes/ as :
Here $f$ is the yield function, so in our case:

$$f = \sigma_{vm}^2 - A_0 - A_1p - A_2p^2$$

(109)

The Föppl symbol is used to indicate that the (visco)plastic strain rates are zero as long as the state of stress is elastic. It should be emphasized at this point that the coefficients of the yield function $f$ are not rate dependent. Phi must be a dimensionless function and tau is a relaxation time.

As an example we assume an exponential law for the rate dependency and a von Mises yield condition, we then obtain after normalisation:

$$\dot{\lambda} = \frac{1}{\tau} \left[ \dot{f}^n \right] = \frac{1}{\tau} \left[ \left( \frac{\sigma_{vm}}{\sigma_y} - 1 \right)^n \right]$$

(110)

If we assume that the state of stress is plastic this gives a two-parameter law for the stress as a function of the plastic strain rate of the Cowper-Symonds type:

$$\sigma_{vm} = \sigma_y \left( 1 + \left( \frac{\tau \dot{\lambda}}{\sigma_y} \right)^n \right)$$

(111)

And the relaxation time is directly related to the reference strain rate:

$$c = \frac{1}{\tau}$$

(112)

The equations above illustrate how the function Phi accounts for the nonlinear aspect of the viscous overstress. It is pointed out in /Hughes/ that the cutting plane underlies a stability condition when applied to rate dependent materials:

$$\Delta t_{n+1} = t_{n+1} - t_n \leq \tau \dot{\lambda}_n$$

(113)

For explicit calculations, this condition should always be fulfilled since the globally controlled timestep will be small compared to the stress relaxation timeframe of the polymer. Indeed for a vonMises type yield surface with linear hardening, associated plasticity and linear viscosity, the above condition becomes:

$$\Delta t \leq \frac{\sigma_y \tau}{3G + H}$$

(114)

This shows that in the case of steel, a timestep of the order of a microsecond will automatically guarantee stability if the relaxation time (tau) is of the order of a few milliseconds. For plastics, a value of tau of a few tenths of a millisecond should be sufficient.
In general Phi need not be dimensionless and in /Wriggers/ the visco-plastic constitutive law is given as:

\[ \dot{\varepsilon}_p = \frac{\langle \Phi(f) \rangle}{2\eta} \Rightarrow \dot{\lambda} = \frac{\langle \Phi(f) \rangle}{2\eta} \]  (115)

Here eta has the dimension of a viscosity if Phi has the dimension of a stress. This formulation is typically based on a constant viscosity and either a power law or exponential expression for Phi and seems usually sufficient to describe the rate dependency of metals.

In the case of plastics, the rate dependency of the material is more pronounced and potentially has a different character. Therefore a tabulated formulation was chosen which gives the user full flexibility for fitting the model to test data. Uniaxial dynamic tensile tests allow tabulating the dynamic stress as a function of plastic strain and plastic strain rate. In the visco-plastic regime we can state:

\[ 2\eta\dot{\lambda} = \Phi(f) \Rightarrow f = \Phi^{-1}(2\eta\dot{\lambda}) \]  (116)

In our implementation we will solve the constitutive equation as follows:

\[ f(\lambda) - \Phi^{-1}(\dot{\lambda}) = 0 \]  (117)

Showing that the viscous overstress expressed by a positive value of the yield function f can be identified with the inverse function of Phi. In contrast to the rate independent case we now enforce the constitutive law instead of the consistency equation. We apply the cutting plane algorithm accordingly.

We briefly summarize the formulation below. It is important to test for plasticity using the usual (rate-independent) yield condition, if the trial stresses are above the yield surface then the viscoplastic constitutive law is enforced:

\[ f = f(\lambda, \dot{\lambda}) \]
\[ f^0_{n+1} = f(\lambda_{n+1}, 0) \]
\[ f^0_{n+1} > 0 \Rightarrow f_{n+1} = \Phi^{-1}(2\eta\dot{\lambda}_{n+1}) \]  (118)

Discretization yields:

\[ \lambda_{n+1}^k = \lambda_{n+1}^k + \Delta \lambda_{n+1}^k \]
\[ \dot{\lambda}_{n+1}^k = \dot{\lambda}_{n+1}^k + \frac{\Delta \lambda_{n+1}^k}{\Delta t} \]
\[ f_{n+1}^{k+1} = f_{n+1}^k + \left( \frac{\partial f}{\partial \sigma} \right)^k_{n+1} \Delta \lambda_{n+1}^k + \left( \frac{\partial f}{\partial q} \right)^k_{n+1} \Delta \lambda_{n+1}^k = \Phi^{-1}(2\eta\dot{\lambda}_{n+1}^k) + \frac{\partial \Phi^{-1}}{\partial \lambda} \frac{\Delta \lambda_{n+1}^k}{\Delta t} \]  (119)

The individual gradients are defined exactly as in (88) and (89):
The derivative of the inverse of Phi is estimated in the numerical algorithm by computing rate dependent values of the hardening parameters:

\[
\frac{\partial \Phi^{-1}}{\partial \lambda} = -\left( \frac{\partial f}{\partial q} \right)_{n+1}^k \left( \frac{\partial q}{\partial \lambda} \right)_{n+1}^k
\]

(121)

And we also introduce a new function h-prime:

\[
\left( \frac{\partial q}{\partial \lambda} \right)_{n+1}^k + \left( \frac{\partial q}{\partial \lambda} \right)_{n+1}^k \frac{1}{\Delta t} = h_{n+1}^k
\]

(122)

Bringing all terms to the left hand side leads to:

\[
f_{n+1}^k - \Phi^{-1}(2\eta^k \lambda_{n+1}^k) - \left( \frac{\partial f}{\partial q} \right)_{n+1}^k c_{n+1}^k \lambda_{n+1}^k \Delta \lambda_{n+1}^k + \left( \frac{\partial f}{\partial q} \right)_{n+1}^k h_{n+1}^k \Delta \lambda_{n+1}^k = 0
\]

(123)

The update for the plastic consistency parameter that is then form equivalent to (90):

\[
\Delta \lambda_{n+1}^{(k)} = \frac{f_{n+1}^{(k)} - \Phi^{-1}(2\eta^{(k)} \lambda_{n+1}^{(k)})}{\left( \frac{\partial f}{\partial q} \right)_{n+1}^{(k)} c_{n+1}^{(k)} \lambda_{n+1}^{(k)} \Delta \lambda_{n+1}^{(k)} + \left( \frac{\partial f}{\partial q} \right)_{n+1}^{(k)} h_{n+1}^{(k)} \Delta \lambda_{n+1}^{(k)}}
\]

(124)

The terms in the nominator can be interpreted as follows in the case of SAMP:

\[
f = \sigma_{\text{vm}}^2 - A_0 (\lambda, \dot{\lambda}) - A_1 (\lambda, \dot{\lambda}) p - A_2 (\lambda, \dot{\lambda}) p^2 \leq 0 \quad \text{consistency condition}
\]

\[
\sigma_{\text{vm}}^2 - A_0 (\lambda, \dot{\lambda}) - A_1 (\lambda, \dot{\lambda}) p - A_2 (\lambda, \dot{\lambda}) p^2 = 0 \quad \text{constitutive law}
\]

\[
\Phi^{-1}(2\eta \lambda) = A_0 (\lambda, \dot{\lambda}) + A_1 (\lambda, \dot{\lambda}) p + A_2 (\lambda, \dot{\lambda}) p^2 - A_0 (\lambda, \dot{\lambda}) + A_1 (\lambda, \dot{\lambda}) p - A_2 (\lambda, \dot{\lambda}) p^2
\]

(125)

The algorithmic changes with respect to the rate-independent case are in the evaluation of rate dependent hardening parameters for the yield function and their derivatives:

\[
\frac{\partial A_0}{\partial \lambda} = \left( \frac{\partial A_0}{\partial \lambda} + \frac{\partial A_0}{\partial \lambda} \frac{1}{\Delta t} \right)
\]

\[
\frac{\partial A_1}{\partial \lambda} = \left( \frac{\partial A_1}{\partial \lambda} + \frac{\partial A_1}{\partial \lambda} \frac{1}{\Delta t} \right)
\]

\[
\frac{\partial A_2}{\partial \lambda} = \left( \frac{\partial A_2}{\partial \lambda} + \frac{\partial A_2}{\partial \lambda} \frac{1}{\Delta t} \right)
\]

(126)
With of course as before:

\[
\frac{\partial A_0}{\partial \lambda} = \frac{A_0}{\lambda} + \frac{\partial A_0}{\partial \sigma_y} \frac{\partial \sigma_y}{\partial \lambda} + \frac{\partial A_0}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial \lambda} + \frac{\partial A_0}{\partial \sigma_c} \frac{\partial \sigma_c}{\partial \lambda} \]

\[
\frac{\partial A_0}{\partial \lambda} = 6\sigma_s \frac{\partial \sigma_s}{\partial \lambda}
\]

And the necessary table-lookups must be performed as described in 2.3 and 2.4. It just needs to be specified that:

\[
\frac{\partial \sigma_s}{\partial \lambda} = \frac{\sigma_s(\lambda_{n+1}^k, \lambda_{n+1}^h) - \sigma_s(\lambda_{n+1}^k, \lambda_{n+1}^h)}{\lambda_{n+1}^k - \lambda_{n+1}^h}
\]

\[
\frac{\partial \sigma_s}{\partial \lambda} = \frac{\sigma_s(\lambda_{n+1}^k, \lambda_{n+1}^h) - \sigma_s(\lambda_{n+1}^k, \lambda_{n+1}^h)}{\lambda_{n+1}^k - \lambda_{n+1}^h}
\]

Just check that this formulation leads to the familiar expression in the radial return algorithm for von-Mises plasticity with linear viscoplasticity and linear hardening:

\[
\Delta \lambda = \frac{\sigma_{\text{vm},e,n+1} - A_{2,n} P_{e,n+1} - A_{1,n} P_{e,n+1} - A_{2,n}}{3G \sigma_{\text{vm},e,n+1} A_{e,n+1}^2 + P_{e,n+1} + \frac{\partial A_2}{\partial \lambda} + \frac{\partial A_1}{\partial \lambda} + \frac{\partial A_0}{\partial \lambda} + \alpha K \frac{\partial p_{e,n+1}}{g_{e,n+1} + A_{1,n} + 2A_{2,n} P_{e,n+1}}}
\]

\[
\Delta \lambda = \frac{\sigma_{\text{vm},e,n+1} - A_{0,n}}{3G + \frac{\partial A_0}{\partial \lambda} + \frac{\partial A_0}{\partial \lambda} + \frac{\partial A_0}{\partial \lambda} \Delta t} = \frac{\sigma_{\text{vm},e,n+1} - \sigma_{y,n}}{3G + H + \Delta t}
\]

3 Applications

3.1 Yield surfaces of different thermoplastics

In a first example, we show the application of the presented model due to prediction of yield for different thermoplastics. The results are published in the thesis of Vogler [23]. For a state-of-the-art review, see the thesis of Koesters [24], the results obtained by SAMP are compared to the von Mises yield criterion. We have to emphasize that this criterion is usually used in crashsimulation for modelling thermoplastics. The experimental results are taken from Bardenheier [7]. For a better understanding, the curves given in Figure 17 to Figure 22 are plotted in both the plane stress plane and the invariant plane, also known as p-q-plane or Burzynski-plane. Experimental results of PVC an abbreviation for
polyvinyl chloride is depicted in Figure 17. The dotted line represents results of using von Mises yield surface and the solid line represents results obtained by SAMP. As can be seen, the von Mises yield surface is not capable to consider the different behaviour under compression, tension and shear. SAMP yields to a much better agreement with the test. However, the experimental result under biaxial tension is not fitted exactly but, from an engineering point of view, approximated sufficiently.

As a next example for polymers widely used in engineering practice, we consider polystyrene (PS). For this polymeric material, more experimental results under different loading direction are available, see Figure 18. Again, von Mises model cannot describe the complicated material response. The results obtained by SAMP are in a pretty good agreement with the experimental findings which makes that model a first class one for choosing an appropriate material law for polymers in future. Similar results can be observed for a polycarbonate in Figure 19. Note that for polystyrene and polycarbonate a convex yield surface is obtained in p-q-plane. A detailed discussion on convexity is given in appendix 2 of this paper.

Further examples showing a nice fitting to experimental results under tension compression and shear are given in Figure 20 for polypropylene (PP) and Figure 21 for polyethylene (PE). We have to mention, however, that similar to PVC a deviation due to biaxial loading cannot be excluded. This discrepancy is usually associated with the phenomenon of crazing which requires a special numerical treatment.
Figure 19: Yield surface of polycarbonate (PC)

Figure 20: Yield surface of polypropylene (PP)

Figure 21: Yield surface of polyethylene (PE)
Such a numerical treatment can be achieved approximately by considering the experimental data under biaxial tension as well as the data under tension, compression and shear. Since we have a quadratic yield surface, a least square fit of the available four data points yields to a pretty good approximation in Figure 22. Here, an acrylonitrile butadiene styrene (ABS) shows noticeable crazing in form of a white fracture surface. In the yield surface, this results in a softening behaviour under biaxial tension. As can be seen, using compression, tension and shear only, the yield surface cannot be described sufficiently by using SAMP. If biaxial tension is considered additionally by least square fit, the (still convex) yield surface is much closer to the experimental data. Details for using such an approach can be found in appendix 1.

3.2 Verification and validation of PP-EPDM

3.2.1 Quasi-static tensile tests with unloading

The most stumbling block in the simulation of thermoplastics is the prediction of elastic rebound in structures. Thereby, the unloading behaviour is determined by viscoelastic effects, i.e. viscosity below the yield surface. Such viscoelasticity is not considered within SAMP so far. However, unloading can be approximated linearly by applying damage to Young’s modulus, see [10], [11] and [12] for details of this modelling technique. For identification, unloading tests at different strain levels have been performed by the Ernst Mach Institute in Freiburg. The unloading path can then be approximated by an effective Young’s modulus which determines the damage parameter \( d = d(\varepsilon_p) \) in dependence of the plastic strain.
In SAMP, this damage function can be tabulated in a load curve LCID-d, see appendix 1. Since the damage model by Lemaitre and Chaboche [20] is used, the yield function \( \sigma_y (\varepsilon_p) \) is effected by damage too, i.e. a softening of the stress-strain curve may be expected. Usually, if elastic damage shall be taken into account only, the yield stress has to be modified according to \( \sigma_y \rightarrow \sigma_y \left( 1 - d \right) \) like it is known from MAT_PLASTICITY_WITH_DAMAGE (material no. 81) in LS-DYNA. To avoid such a cumbersome procedure, we have implemented a useful tool: If LCID-d is negative in the input, the modification of the yield stress is realized internally and the original stress-strain curve is reflected. The simulation results of a simple tensile test in comparison with experimental results are depicted in Figure 23. As can be seen, the reduction of the elastic parameters for increasing plastic strains considers the unloading behaviour approximately. Finally, the experimental stress-strain curve is recovered.

### 3.2.2 Dynamic tensile tests

Now, we demonstrate the strain rate dependency of SAMP by the simulation of dynamic tensile tests. The experimental setup consists of a dogbone specimen with a total length of 48.6mm, see Figure 24. The area of uniaxial stress (8x20mm) is highlighted in the picture. All experiments have been performed, again, by the Ernst Mach Institute in Freiburg. For this load case, SAMP can be treated in the same and comfortable way like the well known from MAT_PIECEWISE_LINEAR_PLASTICITY (material no. 24) in LS-DYNA: local stress-strain curves measured at different strain rate levels are directly input in the material card. The dogbone specimen is then used in simulation and the global force and displacement are compared to the experimental response. The results are given in Figure 24. At this point, it has to be emphasized again that in SAMP a real visco-plastic formulation is implemented, i.e. relaxation effects are taken into account as it is known from MAT_PIECEWISE_LINEAR_PLASTICITY with VP=1. An unphysical formulation like VP=0 is not implemented in SAMP.

![Figure 24: Dynamic tensile tests at different strain rates](image)

### 3.2.3 Compression test

The different behaviour of thermoplastics under compression and tension does not accept a von Mises type of plasticity like it is used in MAT_PIECEWISE_LINEAR_PLASTICITY. In SAMP, the direct input of experimental data obtained from compression tests allows a straightforward treatment of the problem. In Figure 25, a draft of the experimental setup consisting of a dogbone specimen with a total length of 176.33mm is given. The area (40x60mm) of uniaxial stress is highlighted and, additionally, the dogbone specimen is bounded perpendicular to drawing plane to avoid local buckling. The experiments are performed at the DKI in Darmstadt. The dogbone specimen is used in simulation and the global force and displacement are compared to the measured response. The results given in Figure 25 show a good agreement. Note that the local stress-strain curve obtained in the experiment is simply used in the material card for LCID-c. However, a pure uniaxial stress state is hard to achieve in both experiment and simulation. A certain interaction with tension and shear cannot be avoided completely. This is even more pronounced in the shear test of the next example.
3.2.4 Shear test

In Figure 26, a shear test has been simulated by using SAMP. The tests are performed at the DKI in Darmstadt. Although a pure shear stress is certainly not given, the test can be simulated at least up to yield in a satisfactory way. For larger displacements, the model acts too stiff what is an effect of large elements. The effect is getting less pronounced for a finer mesh.

3.2.5 Bending test

In Figure 27, the reaction force versus the displacement of a quasi-static three point bending test is shown. Because of the higher yield stress under compression, it is not possible to simulate the bending test by using von Mises plasticity based on the tensile test data. With SAMP, where the higher yield stress taken from the experiment under compression is considered, the bending test can be simulated with good agreement.

In conclusion it can be said that all the effects associated with thermoplastics given in the examples can be approximately considered in our material model: necking and strain rate effects by visco-plasticity, unloading behaviour by a (1-d)-damage model, different behaviour under compression and tension and thus a correct bending stiffness by the chosen yield surface.
3.3 Further application

SAMP is not only suitable for thermoplastics or restricted to such nasty things – it’s the egg-laying-wool-milk-pig

Figure 28: Structural foam under different loading direction

4 Conclusion and outlook

As an ever increasing percentage of the weight of an automobile is made out of plastic parts and these parts are safety-relevant, the industrial need for a reliable simulation tool can no longer be ignored. Such a tool is subject to the many constraints imposed by the context of an industrial development process: One has to balance the need for accuracy and respect for the actual physics with easiness of use, robustness and efficiency. For SAMP we have therefore chosen a theoretical and nu-
Numerical framework that is familiar, yet reasonably general. The result is, like any numerical methodology, a compromise.

As limitations of the current implementation we can cite the following:

- The method is completely phenomenological: No micromechanics are considered and no attempt is made to take the actual physical deformation processes of the plastic into account.
- The data input is based on test results that are partly difficult to obtain. The latter is primarily true for shear and biaxial loading but also for compressive loads and for any dynamic load involving stress relaxation processes.
- The application is limited to ductile plastics that are initially isotropic and remain isotropic throughout the deformation process. This will not be the case for most polymeric materials.
- In many cases the test data is hard to fit with a quadratic yield surface or will lead to a concave yield surface. Moreover, non-convex yield surfaces potentially generate problems in the numerical algorithm since a unique solution is no longer guaranteed. This situation can be countered balanced partly by assuming isochoric behaviour.

As positive aspects we shall mention:

- For many plastics we were able to obtain a reasonable fit of the experimental data; even if its rate dependency is high.
- The material law is equipped with special-purpose options to obtain a numerically robust response.
- The tabulated input will be transparent to most users.
- The setup of the user-subroutine is such that future extensions are easily incorporated.

Future work will focus on the load-induced anisotropy which develops naturally in plastics and on the implementation of more advanced damage and failure models. On the somewhat longer term a visco-elastic-visco-plastic approach is envisioned to accommodate rate effects in the elastic region. Combined with the anisotropic formulation this can result in a model suitting also fibre reinforced components.

5 Literature


[22] V. Kolupaev, M. Moneke, N. Darsow: Modelling of the three-dimensional creep behaviour of non-reinforced thermoplastics, Computational Materials Science 32 (3-4) (2005), 400-406.


Appendix 1: User manual for *MAT_SAMP-1

This is Material Type 42 (Semi-Analytical Model for Polymers), currently implemented in LS-DYNA as a user-subroutine. Consequently the first 2 cards in this manual are standard input for user-subroutines. This material model uses an isotropic C-1 smooth yield surface for the description of non-reinforced plastics. Details of the implementation are given in [DuBois, Haufe, Kolling & Feucht 2005].

**Card Format**

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<td>I</td>
<td>I</td>
<td>I</td>
<td>I</td>
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<td>BULK</td>
<td>F</td>
<td>F</td>
<td>F</td>
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<td>Mass density</td>
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<td>Material law type, set to 42 for SAMP-1</td>
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<td>LMC</td>
<td>Number of input variables, set to 32, reflecting 4 additional cards</td>
</tr>
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<td>NHV</td>
<td>Number of history variables, currently 32 history variables are stored, conservatively NHV may be set to 50</td>
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<tr>
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<td>set to zero, material law I isotropic</td>
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<td>bulk modulus address in input vector, set to 1</td>
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<tr>
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<td>shear modulus address in input vector, set to 2</td>
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<td>IEOS</td>
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<td>Bulk modulus, used by LS-DYNA in the time step calculation</td>
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<tr>
<td>GMOD</td>
<td>Shear modulus, used by LS-DYNA in the time step calculation</td>
</tr>
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<td>EMOD</td>
<td>Young’s modulus</td>
</tr>
<tr>
<td>NUE</td>
<td>Poisson ratio</td>
</tr>
<tr>
<td>LCID-T</td>
<td>load curve or table ID giving the yield stress as a function of plastic strain, these curves should be obtained from quasi-static and (optionally) dynamic uniaxial tensile tests, this input is mandatory and the material model will not work unless at least one tensile stress-strain curve is given</td>
</tr>
<tr>
<td>LCID-C</td>
<td>load curve ID giving the yield stress as a function of plastic strain, this curve should be obtained from a quasi-static uniaxial compression test, this input is optional</td>
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<td>LCID-S</td>
<td>load curve ID giving the yield stress as a function of plastic strain, this curve should be obtained from a quasi-static shear test, this input is optional</td>
</tr>
<tr>
<td>LCID-B</td>
<td>load curve ID giving the yield stress as a function of plastic strain, this curve should be obtained from a quasi-static biaxial tensile test, this input is optional</td>
</tr>
<tr>
<td>NUEP</td>
<td>plastic Poisson ratio: an estimated ratio of transversal to longitudinal plastic rate of deformation should be given, a value &lt;0 will result in associated plasticity to the yield surface (the associated plasticity option is implemented only for IQUAD=1)</td>
</tr>
<tr>
<td>LCID-P</td>
<td>load curve ID giving the plastic Poisson ratio as a function of equivalent plastic deformation during uniaxial tensile testing, if the (optional) load curve is given, the constant value in the previous field will be ignored</td>
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</tbody>
</table>
| LCID-D   | load curve ID giving the damage parameter as a function of equivalent plastic deformation during uniaxial tensile testing, by default this option assumes that effective yield values are used in the load curves LCID-T, LCID-C, LCID-S and LCID-B, if LCID-D is given a nega-
tive value, true (original) yield stress values are used, i.e. yield stress is divided by \((1-d)\) internally

**DC**
critical damage value at failure if LCID_D is defined, otherwise, (if LCID_D=0) this parameter is the equivalent plastic strain at failure (EPFAIL)

**DEPRPT**
increment of equivalent plastic strain between failure point and rupture point, stresses will fade out to zero between EPFAIL and EPFAIL+DEPRUPT

**IDUM_C**
dummy index for load curve LCID_C:
- IDUM_C=0: load curve contains real data
- IDUM_C=1: load curve contains dummy data

**IDUM_S**
dummy index for load curve LCID_S:
- IDUM_S=0: load curve contains real data
- IDUM_S=1: load curve contains dummy data

**IDUM_B**
dummy index for load curve LCID_B:
- IDUM_B=0: load curve contains real data
- IDUM_B=1: load curve contains dummy data

**MITER**
maximum number of iterations in the cutting plane algorithm, default is set to 400

**MIPS**
maximum number of iterations in the secant iteration performed to enforce plane stress (shell elements only), default set to 10

**IVM**
formulation flag:
- IVM=0: SAMP-1 formulation (default)
- IVM=1: vonMises associated plasticity

**IQUAD**
formulation flag:
- IQUAD=0: yield surface linear in the vonMises stress (default)
- IQUAD=1: yield surface quadratic in the vonMises stress (recommended)

**ICONV**
formulation flag:
- ICONV=0: default
- ICONV=1: yield surface is internally modified by increasing the shear yield until a convex yield surface is achieved
- ICONV=2: if the plastic Poisson ratio is smaller than the elastic Poisson ratio, both are set equal to the smaller value of the two

**ASAF**
safety factor, used only if ICONV=1, values between 1 and 2 can improve convergence, however the shear yield will be artificially increased if this option is used, default is set to 1.
Remarks:

Due to the current implementation as a user subroutine it is necessary to input a load curve for the material yield under compressive and shear loads even if no data are available. Only the load curve number for the biaxial characteristic may be equal to zero. If dummy load curves are provided they need to have the same number of data points as the load curve LCID_T giving the material yield stress under tensile loading. If multiple materials are defined, different dummy curves should be defined for each material.

The dummy curves can contain any data since reasonable values will be generated internally as follows:

\[
\begin{align*}
\sigma_c &= \sigma_t \\
\sigma_s &= \frac{\sigma_t}{\sqrt{3}}
\end{align*}
\]

\[
\begin{align*}
\sigma_c &= \frac{\sqrt{3} \sigma_t \sigma_s}{2 \sigma_t - \sqrt{3} \sigma_s} \\
\sigma_s &= \frac{\sqrt{3} \sigma_t + \sigma_c}{\sqrt{3}}
\end{align*}
\]

\[
\begin{align*}
\sigma_c &= \frac{\sigma_t \sigma_b}{3 \sigma_t - 2 \sigma_b} \\
\sigma_s &= \frac{\sigma_s \sigma_b}{\sqrt{3}(2 \sigma_b - \sigma_t)}
\end{align*}
\]

A linear yield surface in the invariant space spanned by the pressure and the von Mises stress is generated using the available data points.
If more then 2 load curves are available the following cases can be distinguished:

\[
\begin{align*}
IDUM\_C &= 0 \\
IDUM\_S &= 0 \\ IDUM\_B &= 1 \quad \Rightarrow SAMP - 1 \\
IDUM\_C &= 0 \\
IDUM\_S &= 1 \\ IDUM\_B &= 0 \quad \Rightarrow \sigma_s = \frac{1}{\sqrt{3}} \sqrt{3\sigma_i^2 \sigma_j \sigma_l} \\
IDUM\_C &= 1 \\
IDUM\_S &= 0 \\ IDUM\_B &= 0 \quad \Rightarrow \sigma_c = \frac{6 \left( 16 \sigma_i^2 \sigma_j^2 + \sigma_i \sigma_j \sigma_l \right)}{6\sigma_i \sigma_j^2 + 32 \sigma_i \sigma_j \sigma_l + 3 \sigma_j^2 \sigma_l}
\end{align*}
\]

If the LCID_D is given, a damage curve as a function of equivalent plastic strains acting on the stresses is defined as depicted in the following picture. DC and DEPRPT defined the failure and fading behaviour of a single element. If LCID_D is positive, both Young’s modulus and yield surface is damaged according to the damage curve. If LCID_D is negative, Young’s modulus is damaged only.

**Appendix 2: Convexity of the SAMP-1 yield surface**

To study convexity in stress space it will be sufficient to consider a reduced form of the yield function containing only the normal stresses:

\[
f = \sigma^\mathbf{F} \sigma + \mathbf{B}^\mathbf{F} \sigma + F_0 \leq 0
\]  

(130)
\[ \sigma = \begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \end{pmatrix}, \quad F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}, \quad B = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \]

The eigenvalues of \( F \) are:

\[ R^T FR = \begin{pmatrix} F_{11} - F_{12} & 0 & 0 \\ 0 & F_{11} - F_{12} & 0 \\ 0 & 0 & F_{11} + 2F_{12} \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 1 & \sqrt{3} \\ -1 & \sqrt{2} & 0 \\ 0 & \sqrt{2} & 1 \end{pmatrix} \]

The yield surface can now be expressed as:

\[ f = \sigma^T R^T F R \sigma + B^T R \sigma + F_0 \leq 0 \]

\[ f = (F_{11} - F_{12})^2 (\sigma_{11}^2 + \sigma_{22}^2) + (F_{11} + 2F_{12})^2 \sigma_{33}^2 + \sqrt{3} F_1 \sigma_{33} + F_0 \]

\[ f = 6(\sigma_{11}^2 + \sigma_{22}^2) - \frac{A_2}{3} \sigma_{33}^2 + \frac{A_1}{\sqrt{3}} \sigma_{33} - A_0 \]

Where we know that:

\[ F_0 \leq 0 \]

\[ F_{11} - F_{12} = 2 F_{44} = 6 > 0 \]

Just assuming that no negative values are given for the yield stress in any experiment. Consequently with respect to the eigenvalues we can examine the following cases:

\[ F_{11} - F_{12} > 0 \quad F_{11} + 2F_{12} > 0 \quad \text{convex} \]

\[ F_{11} - F_{12} > 0 \quad F_{11} + 2F_{12} < 0 \quad \text{concave} \]

\[ F_{11} - F_{12} > 0 \quad F_{11} + 2F_{12} = 0 \quad \text{convex} \]

In the first case we now consider that all eigenvalues are non-zero. In this case we perform another transformation of the reference system, this time a translation:

\[ \begin{pmatrix} \sigma_{aa} \\ \sigma_{bb} \\ \sigma_{cc} \end{pmatrix} = \begin{pmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \frac{F_1 \sqrt{3}}{2(F_{11} + 2F_{12})} \end{pmatrix} \]

The yield surface can now be expressed as:
\[ f = 0 \Rightarrow (F_{11} - F_{12}) \left( \sigma_{aa}^2 + \sigma_{bb}^2 \right) + (F_{11} + 2F_{12}) \sigma_{cc}^2 = -F_0 + \frac{3F_1^2}{4(F_{11} + 2F_{12})} \] (137)

We can now distinguish two subcases, if all eigenvalues are positive then:

\[ F_{11} + 2F_{12} > 0 \]

\[ f = 0 \Rightarrow \frac{(F_{11} - F_{12})}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \left( \sigma_{aa}^2 + \sigma_{bb}^2 \right) + \frac{4(F_{11} + 2F_{12})^2}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \sigma_{cc}^2 = 1 \] (138)

And the yield surface is a real ellipsoid and thus convex (since all three coefficients are guaranteed positive). If the third eigenvalue is negative, different possibilities must be considered:

\[ F_{11} + 2F_{12} < 0 \]

\[ 3F_1^2 - 4F_0(F_{11} + 2F_{12}) < 0 \]

\[ f = 0 \Rightarrow \frac{(F_{11} - F_{12})}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \left( \sigma_{aa}^2 + \sigma_{bb}^2 \right) - \frac{4(F_{11} + 2F_{12})^2}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \sigma_{cc}^2 = 1 \] (139)

And the yield surface is a single bladed hyperboloid. Note that if the coefficient \( F_1 \) is equal to zero, the yield surface is always single bladed if it is hyperbolic.

\[ F_{11} + 2F_{12} < 0 \]

\[ 3F_1^2 - 4F_0(F_{11} + 2F_{12}) > 0 \]

\[ f = 0 \Rightarrow \frac{(F_{11} - F_{12})}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \left( \sigma_{aa}^2 + \sigma_{bb}^2 \right) - \frac{4(F_{11} + 2F_{12})^2}{3F_1^2 - 4F_0(F_{11} + 2F_{12})} \sigma_{cc}^2 = -1 \] (140)

And the yield surface is a two-bladed hyperboloid.

\[ F_{11} + 2F_{12} < 0 \]

\[ 3F_1^2 - 4F_0(F_{11} + 2F_{12}) = 0 \]

\[ f = 0 \Rightarrow (F_{11} - F_{12}) \left( \sigma_{aa}^2 + \sigma_{bb}^2 \right) - (F_{11} + 2F_{12}) \sigma_{cc}^2 = 0 \] (141)

And the yield surface is a real cone. This last case obviously corresponds to a Drucker-Prager type yield surface as can easily be established.

\[ 3F_1^2 - 4F_0(F_{11} + 2F_{12}) = 0 \] (142)

And

\[ F_0 = -A_0, F_1 = \frac{A_1}{3}, F_{11} = 1 - \frac{A_2}{9}, F_{44} = 3, F_{12} = F_{11} - \frac{F_{44}}{2} = -\left( \frac{1}{2} + \frac{A_2}{9} \right) \] (143)

Lead to
\[
\frac{A_i^2}{3} + 4A_0 \left( -\frac{3A_2}{9} \right) = 0 \quad \Rightarrow \quad A_i = 2\sqrt{A_2A_0}
\] (144)

Which is the condition for the general quadratic yield surface to correspond to a straight line in the invariant plane.

In the second case we assume that one of the three eigenvalues is zero, this can only be the third eigenvalue corresponding to the hydrostatic axis:

\[
f = \sigma^T R^T F R \sigma + R^T B R + F_0 \leq 0
\]
\[
F_{11} + 2F_{12} = 0
\]
\[
f = 0 \quad \Rightarrow \quad (F_{11} - F_{12}) (\sigma_{11}^2 + \sigma_{22}^2) = -\sqrt{3}F_1 \sigma_{33} - F_0
\]

Is also the coefficient $F_1$ equal to zero then:

\[
f = 0 \quad \Rightarrow \quad (F_{11} - F_{12}) (\sigma_{11}^2 + \sigma_{22}^2) = 1
\] (146)

And the yield surface corresponds to a real elliptic cylinder. This is equivalent to the vonMises case and, of course, this yield surface is also convex. If on the other hand the coefficient $F_1$ is not equal to zero we perform another translation of the reference system:

\[
\begin{pmatrix}
\sigma_{aa} \\
\sigma_{bb} \\
\sigma_{cc}
\end{pmatrix}
=
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33}
\end{pmatrix}
+
\begin{pmatrix}
0 \\
0 \\
F_0
\end{pmatrix}
\begin{pmatrix}
0 \\
\sqrt{3}F_1
\end{pmatrix}
\] (147)

And the yield surface becomes

\[
f = 0 \quad \Rightarrow \quad (F_{11} - F_{12}) \frac{2}{-\sqrt{3}F_1} (\sigma_{aa}^2 + \sigma_{bb}^2) = 2\sigma_{cc}^*
\] (148)

which gives an elliptical paraboloid and is again convex.