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Selective mass scaling for explicit finite element analyses

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SUMMARY

Due to their inherent lack of convergence problems explicit finite element techniques are widely used for analysing non-linear mechanical processes. In many such processes the energy content in the high frequency domain is small. By focusing an artificial mass scaling on this domain, the critical time step may be increased substantially without significantly affecting the low frequency behaviour. This is what we refer to as selective mass scaling. Two methods for selective mass scaling are introduced in this work. The proposed methods are based on non-diagonal mass matrices that scale down the eigenfrequencies of the system. The applicability of the methods is illustrated in two example models where the critical time step is increased by up to 30 times its original size. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: selective mass scaling; finite element; explicit time integration

1. INTRODUCTION

Simulation based design in general and the finite element method in particular has today become a standard tool in industry, by which expensive and time consuming experiments to a large extent have been eliminated.

Considering the simulation of rapid transient phenomena such as for example vehicle impacts, one finds that explicit time integration is the dominating procedure. This is due to its inherent lack of convergence problems, and to the fact that the time step size needed to resolve the transients is often of the same order of magnitude as the critical time step imposed by the conditional stability of the time integration scheme.

Explicit time integration has also been found extremely powerful in simulating some less rapid processes, such as for example sheet metal forming operations. The clue to its success in these applications is that a moderate artificial increase in the loading rate and/or a moderate artificial mass scaling do not significantly change the response of the system. Thus, by using such 'tricks', realistic solution times may be achieved. However, it has to be noted that a too

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severe scaling cannot be applied, since it unavoidably will introduce unwanted, non-physical, inertia effects in the structure under consideration.

In finite element models of less rapid processes, most kinetic energy generally resides in the lower eigenfrequency domain, while only less significant fractions are contained in activated high frequency modes. With this as a basis, one may ask if the blunt procedure of ordinary mass scaling for explicit time integration schemes may be replaced by a more precise selective mass scaling procedure.

Since the critical time step is limited by the highest eigenfrequency of the system, it would be preferable to focus the mass scaling to the upper eigenfrequency domain which generally only contains small amounts of kinetic energy. The lower eigenfrequency domain should remain as unaffected as possible. This is what we refer to as selective mass scaling. Olovsson *et al.* [1] suggest a method for selective mass scaling by acceleration filtering, that is valid for thin walled structures modelled with solid elements. As a continuation of that work, two more general methods for selective mass scaling are introduced in this article. The methods are more general in the sense that they are valid for any displacement based finite element formulation and they are not limited to the modelling of thin walled structures. Both methods are based on a technique where additional terms are added to the mass matrix.

The disposition of the article is as follows: A brief introduction to the dynamics of a discretized system in Section 2 is followed by a description of the suggested methods for selective mass scaling in Section 3. Finally, two numerical application examples are given in Section 4.

2. DYNAMICS OF A DISCRETIZED SYSTEM

Assume an undamped and spatially discretized system with the tangential stiffness matrix $\mathbf{K}$ and the mass matrix $\mathbf{M}$.

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix}$$
$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{aa} & \mathbf{M}_{ab} \\ \mathbf{M}_{ba} & \mathbf{M}_{bb} \end{bmatrix} \tag{1}$$

such that

$$\begin{bmatrix} \mathbf{M}_{aa} & \mathbf{M}_{ab} \\ \mathbf{M}_{ba} & \mathbf{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}}_a \\ \ddot{\mathbf{u}}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_a^{\text{ext}} \\ \mathbf{F}_b^{\text{ext}} \end{Bmatrix} - \begin{Bmatrix} \mathbf{F}_a^{\text{int}} \\ \mathbf{F}_b^{\text{int}} \end{Bmatrix} \tag{2}$$

and

$$\begin{Bmatrix} \dot{\mathbf{F}}_a^{\text{int}} \\ \dot{\mathbf{F}}_b^{\text{int}} \end{Bmatrix} = \begin{bmatrix} \mathbf{K}_{aa} & \mathbf{K}_{ab} \\ \mathbf{K}_{ba} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}}_a \\ \dot{\mathbf{u}}_b \end{Bmatrix} \tag{3}$$

where $\mathbf{u}_a$ are unknown node displacements, $\mathbf{u}_b$ are prescribed node displacements, $\mathbf{F}_a^{\text{ext}}$ are prescribed node forces and $\mathbf{F}_b^{\text{ext}}$ are unknown reaction forces. $\mathbf{F}_a^{\text{int}}$ and $\mathbf{F}_b^{\text{int}}$ are internal node forces.

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2.1. Central difference time integration

Using the second-order accurate central difference method for the time integration, c.f. Hughes [2], the current acceleration $\ddot{\mathbf{u}}_a$ is calculated every time step as

$$\ddot{\mathbf{u}}_a = \mathbf{M}_{aa}^{-1}\left(\mathbf{F}_a^{\text{ext}} - \mathbf{F}_a^{\text{int}} - \mathbf{M}_{ab}\ddot{\mathbf{u}}_b\right) \tag{4}$$

$\mathbf{M}$ is generally a lumped, diagonal, matrix such that $\mathbf{M}_{ab} \equiv \mathbf{0}$. Using the central difference time integration scheme, the critical time step Δt_{c} is limited by the maximum eigenfrequency of the system, $\omega_{\max}$, c.f. Hughes [2].

$$\Delta t_{\text{c}} = \frac{2}{\omega_{\max}} \tag{5}$$

The upper bound of the time step size severely limits the usefulness of the central difference time integration scheme, especially when modelling certain quasi-static processes.

2.2. Eigenfrequencies

The linear eigenfrequencies of the system, of which $\omega_{\max}$ is the highest, are obtained by solving the eigenvalue problem

$$\det\left[\mathbf{K}_{aa} - \omega_i^2\mathbf{M}_{aa}\right] = 0 \quad i \in [1, N] \tag{6}$$

where $\mathbf{K}_{aa}$ and $\mathbf{M}_{aa}$ are of the dimension $N \times N$.

3. SELECTIVE MASS SCALING

The basic idea behind this work is to add artificial terms to the mass matrix, aiming at decreasing the highest eigenfrequencies of the system, while affecting the lower frequencies as little as possible.

We will start off on the element level and define a modified element mass matrix $\bar{\mathbf{m}}$.

$$\bar{\mathbf{m}} = \mathbf{m} + \boldsymbol{\lambda} \tag{7}$$

where $\mathbf{m}$ is the original element mass matrix, normally lumped and $\boldsymbol{\lambda}$ contains the artificially added mass terms. The basic philosophy is to define $\boldsymbol{\lambda}$ to lower the non-zero eigenfrequencies of the element, while at the same time not affecting the translational rigid body behaviour at all. That is, for a rigid body translational acceleration $\ddot{\mathbf{u}}_r$, one should have

$$\bar{\mathbf{m}}\ddot{\mathbf{u}}_r = \mathbf{m}\ddot{\mathbf{u}}_r \;\Rightarrow\; \boldsymbol{\lambda}\ddot{\mathbf{u}}_r = \mathbf{0} \tag{8}$$

$\boldsymbol{\lambda}$ can be defined in many different ways and we will take a closer look at two methods. Method I is beneficial for explaining the proposed concept, since it allows for a straightforward analysis of the modified eigenfrequency spectra. Method II is more *ad hoc*, but at the same time better suited for geometrically non-linear applications.

3.1. Method I

Since a rigid body translation does not produce internal forces, Equation (8) will be satisfied by defining

$$\boldsymbol{\lambda} = \alpha\mathbf{k}, \quad \alpha \geqslant 0 \tag{9}$$

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where **k** is the tangential element stiffness matrix. Assembling the modified element mass matrices gives a global mass matrix $\bar{\mathbf{M}}$ according to

$$\bar{\mathbf{M}} = \mathbf{M} + \alpha\mathbf{K} \tag{10}$$

Further, solving the eigenvalue problem in Equation (6) using the modified mass matrix in Equation (10) gives

$$\det\left[\mathbf{K}_{aa} - \bar{\omega}_i^2\bar{\mathbf{M}}_{aa}\right] = 0 \;\Rightarrow\; \det\left[\mathbf{K}_{aa} - \frac{\bar{\omega}_i^2}{1-\alpha\bar{\omega}_i^2}\mathbf{M}_{aa}\right] = 0 \tag{11}$$

That is, the eigenmodes are preserved, but the eigenfrequencies are not. From Equations (6) and (11) it follows that

$$\omega_i^2 = \frac{\bar{\omega}_i^2}{1-\alpha\bar{\omega}_i^2} \;\Rightarrow\; \bar{\omega}_i^2 = \frac{\omega_i^2}{1+\alpha\omega_i^2} \tag{12}$$

Based on the relation in Equation (12), Figure 1 visualizes the effect that different levels of α have on the eigenfrequency spectra. One can see that $\alpha = 100/\omega_{\max}^2$ lowers the maximum eigenfrequency by approximately one order of magnitude, while at the same time leaving the lower frequency domain relatively unaffected. A regular mass scaling, on the other hand, would decrease all eigenfrequencies by the same percentage and, hence, more severely influence the dynamical behaviour of the system.

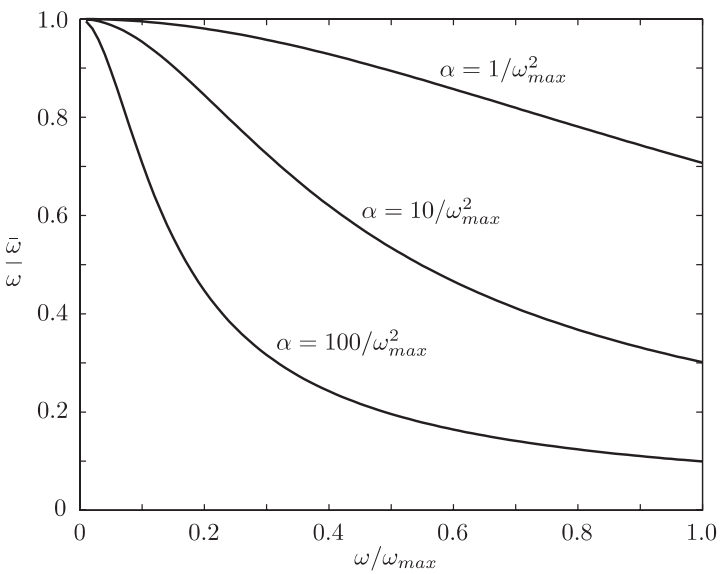

Figure 1. Ratio of modified and original eigenfrequencies at different levels of selective mass scaling using Method I.

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3.2. Method II

According to Equation (4) the mass matrix needs to be inverted for the calculation of the current acceleration. Hence, defining $\boldsymbol{\lambda} = \alpha \mathbf{k}$ is not practical in geometrically non-linear applications, where $\mathbf{k}$ changes throughout the simulation process. In such situations we should look for a $\boldsymbol{\lambda}$ that does not change with element deformation or rotation.

Naturally, $\boldsymbol{\lambda}$ becomes element type dependent and we will focus here on an eight-noded, tri-linear, solid element. However, the ideas can easily be applied to any given displacement based element type.

We first need to define the element local ordering of the 24 displacement degrees of freedom.

$$\mathbf{u}^{\mathrm{e}} = \{u_1 u_2 \cdots u_8 \;\; v_1 v_2 \cdots v_8 \;\; w_1 w_2 \cdots w_8\} \tag{13}$$

$\mathbf{u}^{\mathrm{e}}$ is the element node displacement vector and (u_i, v_i, w_i) are displacements in the x, y, and z-directions of element node i, respectively. Further, we assume a lumped mass matrix $\mathbf{m}$ such that

$$\mathbf{m} = \frac{m^{\mathrm{e}}}{8} \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tag{14}$$

where m^{e} is the total element mass. $\boldsymbol{\lambda}$ is defined to increase the inertia effects for the elastic modes of the element, but not for a rigid translational acceleration.

$$\boldsymbol{\lambda} = \begin{bmatrix} \boldsymbol{\lambda}_{8\times 8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}_{8\times 8} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}_{8\times 8} \end{bmatrix} \tag{15}$$

where

$$\boldsymbol{\lambda}_{8\times 8} = \frac{\beta m^{\mathrm{e}}}{56} \begin{bmatrix} 7 & -1 & \dots & -1 \\ -1 & 7 & & \\ \vdots & & \ddots & \\ -1 & & & 7 \end{bmatrix} \tag{16}$$

$\beta \geqslant 0$ is a parameter controlling the level of selective mass scaling. Note that $\boldsymbol{\lambda}_{8\times 8}$ is defined such that changing β from 0 to 1 doubles the diagonal terms in the mass matrix. Further, $\boldsymbol{\lambda}_{8\times 8}$ is also defined to avoid coupling terms between the global x, y and z-directions. This greatly reduces the computational effort required to invert $\bar{\mathbf{M}}_{aa}$.

To show how β scales the eigenfrequencies, we study a metal strip that has been modelled with fully integrated eight-noded brick elements according to Figure 2. Young's modulus was set to $E = 207$ GPa, Poisson's ratio to $v = 0.3$ and the density to $\rho = 7800\,\mathrm{kg/m^3}$.

Due to shear locking effects, the aspect ratio of the elements is not acceptable. However, this is of less significance in this particular case since the model only serves as an example of how the proposed method for mass scaling influences the eigenfrequencies of a discretized system. Figure 3 displays the original eigenfrequencies of the 1857 degrees of freedom system and

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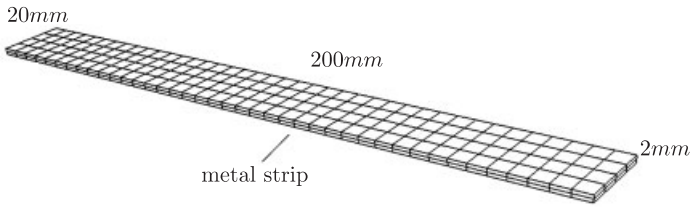

Figure 2. A metal strip modelled with solid elements.

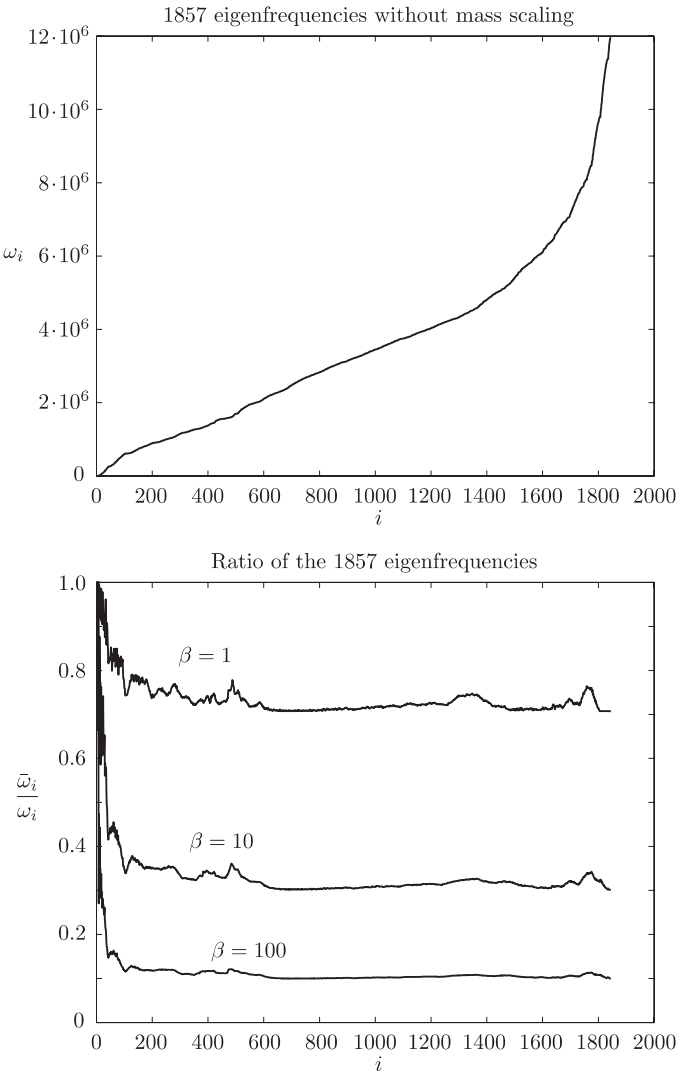

Figure 3. Ratio of eigenfrequencies at different levels of selective mass scaling using Method II.

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the ratio of the modified and original eigenfrequencies at different levels of the mass scaling parameter β. As expected, one can see that the lower eigenfrequencies are less modified by the mass scaling than the higher ones.

4. APPLICATION EXAMPLES

The proposed methods for selective mass scaling have been tested in two different application examples using the central difference time integration scheme. The first one, a linear elastic model of a tip loaded cantilever beam, is well suited for both Methods I and II.

The second example is a three-point bending model where large deformations make Method I too expensive from a computational point of view. We cannot afford to invert $\bar{\mathbf{M}}_{aa}$ every time step.

4.1. Tip loaded cantilever beam

The model of the tip loaded cantilever beam was defined according to Figure 4. Young's modulus, Poisson's ratio and the density were set to $E = 207\,\text{GPa}$, $v = 0.0$ and $\rho = 7800\,\text{kg/m}^3$, respectively.

The tip deflection w was registered using both regular, selective and no mass scaling at all. In absence of mass scaling 12 900 time steps were required to complete 2 ms of physical time. Setting $\beta = 100$ or using regular mass scaling (increasing the density 100 times) cut down the number of steps to less than 1300, while $\beta = 1000$ and $\alpha = 1000/\omega_{\max}$ reduced the number of time steps to approximately 360.

Figure 5 shows the tip deflection as a function of time. Method I is clearly superior to Method II in its ability to increase the critical time step, without significantly altering the dynamical response of the system. At the same time Method II is vastly superior to the use of regular mass scaling. The calculations were carried out using MATLAB.

4.2. Three-point bending of a beam

The second application example is a geometrically non-linear model of an elasto-plastic beam in three-point bending. The model is shown in Figure 6. Young's modulus, Poisson's ratio and the density were set to $E = 207\,\text{GPa}$, $v = 0.3$ and $\rho = 7800\,\text{kg/m}^3$, respectively. A von Mises

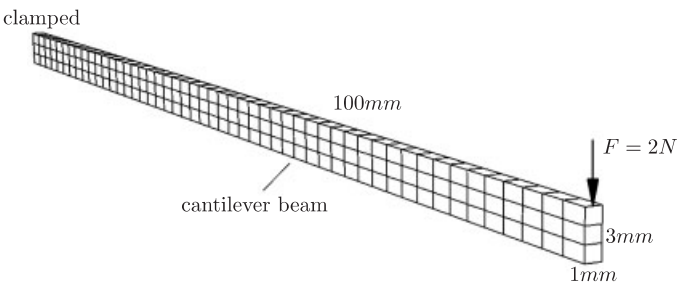

Figure 4. Tip loaded cantilever beam.

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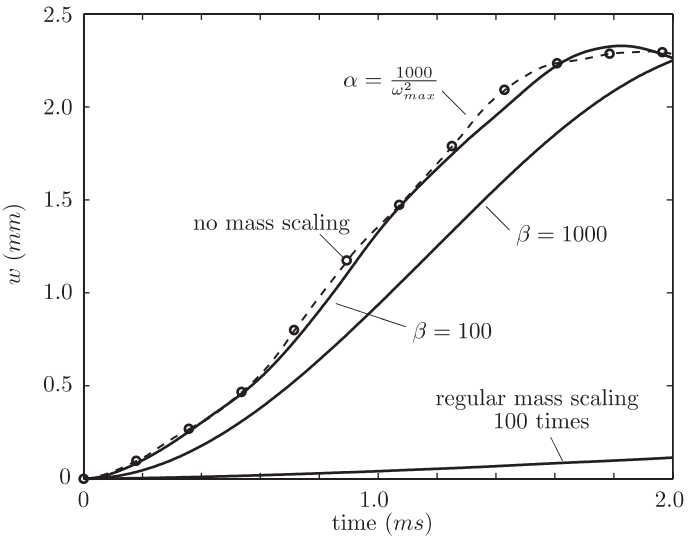

Figure 5. Tip deflection w of a cantilever beam at different levels of mass scaling.

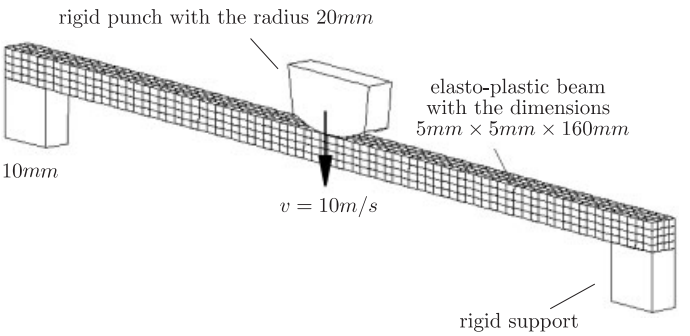

Figure 6. Three-point bending of a beam that is modelled with 1280 elements.

yield criterion was assumed and the yield stress σ_y was defined as a function of the effective plastic strain ε^p_{eff} according to

$$\sigma_y = 400\left(0.03 + \varepsilon^p_{eff}\right)^{0.2} \quad \text{(MPa)} \tag{17}$$

Three simulations were carried out, one without mass scaling, one with regular mass scaling where the density was increased 100 times and one simulation with selective mass scaling where β was set to 1000. The simulations were terminated after 2 ms, as the punch displacement reached 20 mm. Figure 7 shows the response of the beam and the distribution of plastic strains. Using selective mass scaling, Method I, the time step was increased 30 times without significantly altering the dynamical response of the beam.

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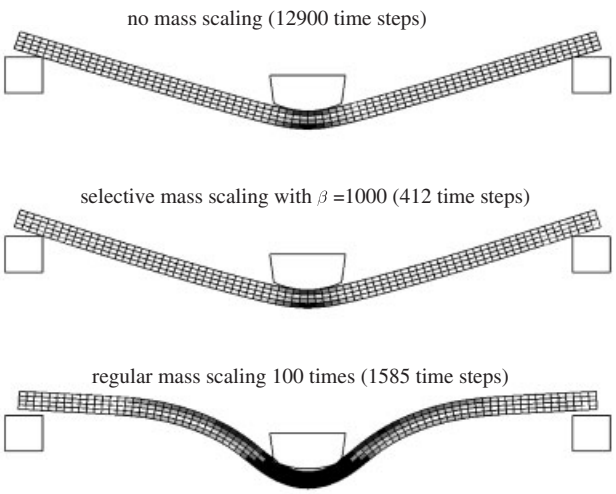

Figure 7. Three-point bending. Black colour indicates effective plastic strains above 7%.

In this application example the explicit finite element code KRYP, c.f. Olovsson [3], was used for the simulations.

5. DISCUSSION

Two different methods for selective mass scaling have been introduced. The methods aim at increasing the critical time step in explicit finite element analyses without significantly altering the dynamical response of the system. Both methods are based on adding artificial, non-diagonal terms to the mass matrix.

Method I is beneficial for explaining the proposed concept, since it allows for a straightforward analytical analysis of the modified eigenfrequency spectra. However, it is not well suited for geometrically non-linear analyses where the mass matrix needs to be inverted several times throughout the simulation process. It remains to be explored whether an approximation to the inverted matrix is good enough for the desired selective mass scaling effect. If this is the case, one might consider inverting the mass matrix less frequently and incorporate Method I for a wider range of applications.

Method II is less precise at distinguishing between high and low frequencies, but at the same time better suited in geometrically non-linear applications, as the modified mass matrix only needs to be calculated and inverted once.

The application examples in Section 4 indicate that selective mass scaling has the potential to greatly reduce the required CPU time when analysing dynamic and quasi-static processes where most kinetic energy resides in the lower frequency domain of the system.

Compared to regular mass scaling, the proposed methods for selective mass scaling require additional memory storage for the inverted mass matrix. However, for large systems, most terms in the inverted matrix are of little significance. Hence, one might consider neglecting some terms to save memory space. Further work is required to explore this possibility.

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