TEST CASE DOCUMENTATION
AND TESTING RESULTS

TEST CASE ID ICFD-VER-9.1

Carreau and Cross flows in a channel

Tested with LS-DYNA® Dev Revision 100276

Friday 28th August, 2015

Document Information	
Confidentiality	**external use**
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Contents

1 Introduction

1.1 Purpose of this Document

This document specifies the test case ICFD-VER-9.1. It provides general test case information like name and ID as well as information to the confidentiality, status, and classification of the test case.

A detailed description of the test case is given, the purpose of the test case is defined, and the tested features are named. Results and observations are stated and discussed. Testing results are provided in section 4.1 for the therein mentioned LS-DYNA® version and platforms.

2 Test Case Information

Test Case Summary	
Confidentiality	external use
Test Case Name	Non Newtonian models : Carreau and Cross flows in a channel
Test Case ID	ICFD-VER-9.1
Test Case Status	Under consideration
Test Case Classification	Verification
Metadata	INTERNAL FLOW

Table 1: Test Case Summary

3 Test Case Specification

3.1 Test Case Purpose

The purpose of this test case is to validate the Carreau and Cross II models for Non-Newtonian flows.

3.2 Test Case Description

The Carreau model is a non-Newtonian fluid model for Pseudo Plastics. For Carreau fluid, the viscosity is given by [2] :

$$\mu = \mu_i + \delta(1 + \lambda^2\gamma^2)^{(n^{'})/2} \tag{1}$$

where μ_i is the infinite shear viscosity, $\delta = (\mu_0 - \mu_i)$ with μ_0 the zero-shear viscosity, $n^{'} = n - 1$ with n the flow behavior index, λ a time constant and γ the shear strain.
The Cross II model is commonly used when it is necessary to describe the low-shear rate behavior of the viscosity. For Cross Fluid, the viscosity is given by [2] :

$$\mu = \mu_i + \frac{\delta}{1 + \lambda^n\gamma^n} \tag{2}$$

In [2], it is shown that an analytical expression exists for those flows in 3D circular tubes and in 2D infinite slits giving the flow rate function of the pressure gradient between the inlet and outlet.
For the Carreau flow, in a 3D circular tube, [2] shows that :

$$Q = \frac{\pi(2L)^3 I}{(\Delta P)^3} \tag{3}$$

and :

$$\left[\mu_i + \delta(1 + \lambda^2\gamma^2)^{n^{'}/2}\right]\gamma = \frac{R\Delta P}{2L} \tag{4}$$

with Q the flow rate, R the radius, I a constant function of γ, L the tube length, τ the local shear stress and ΔP the pressure gradient. γ can be obtained numerically by a simple numerical solver (bisection method works well) using Equation (23). I is then expressed as [2] :

$$I = I1 + I2 + I3 + I4 \tag{5}$$

with :

$$I1 = \frac{\delta^3 \Big[3\lambda^4 (3n'^2 + 5n' + 2)\gamma^4 - 3n'\lambda^2\gamma^2 + 2 \Big] (1 + \lambda^2\gamma^2)^{3n'/2}}{3\lambda^4 (9n'^2 + 18n' + 8)} \tag{6}$$

$$I2 = \frac{\mu_i \delta^2 \Big[\lambda^4 (2n'^2 + 5n' + 3)\gamma^4 - n'\lambda^2\gamma^2 + 1 \Big] (1 + \lambda^2\gamma^2)^{n^1}}{2\lambda^4 (n' + 1)(n' + 2)} \tag{7}$$

$$I3 = \frac{\mu_i^2 \delta \Big[\lambda^4 (2n'^2 + 5n' + 6)\gamma^4 - n'\lambda^2\gamma^2 + 2 \Big] (1 + \lambda^2\gamma^2)^{n^1}}{\lambda^4 (n' + 2)(n' + 4)} + \frac{\mu_i^3 \gamma^4}{4} \tag{8}$$

$$I4 = -\left(\frac{2\delta^3}{3\lambda^4 (9n'^2 + 18n' + 8)} + \frac{\mu_i \delta^2}{2\lambda^4 (n' + 1)(n' + 2)} + \frac{2\mu_i^2 \delta}{\lambda^4 (n' + 2)(n' + 4)} \right) \tag{9}$$

For the Carreau flow, in a 2D infinite slit, we have :

$$Q = \frac{2\pi (L)^2 I}{(\Delta P)^2} \tag{10}$$

and :

$$\Big[\mu_i + \delta (1 + \lambda^2\gamma^2)^{n'/2} \Big] \gamma = \frac{B \Delta P}{L} \tag{11}$$

with B now being half the total height of the slit. I is this time expressed as :

$$I = I1 + I2 + I3 + I4 \tag{12}$$

with :

$$I1 = \frac{n'\delta^2\gamma \Big[{}_2F_1(\tfrac{1}{2}, 1 - n'; \tfrac{3}{2}; -\lambda^2\gamma^2) - {}_2F_1(\tfrac{1}{2}, -n'; \tfrac{3}{2}; -\lambda^2\gamma^2) \Big]}{\lambda^2} \tag{13}$$

$$I2 = \frac{(1 + n')\delta^2\gamma^3 {}_2F_1(\tfrac{3}{2}, -n'; \tfrac{5}{2}; -\lambda^2\gamma^2)}{3} \tag{14}$$

$$I3 = \frac{n'\delta\mu_i\gamma \Big[{}_2F_1(\tfrac{1}{2}, 1 - \tfrac{n'}{2}; \tfrac{3}{2}; -\lambda^2\gamma^2) - {}_2F_1(\tfrac{1}{2}, -\tfrac{n'}{2}; \tfrac{3}{2}; -\lambda^2\gamma^2) \Big]}{\lambda^2} \tag{15}$$

$$I4 = \frac{(2+n')\delta^2\mu_i\gamma^3{}_2F_1(\frac{3}{2}, -\frac{n'}{2}; \frac{5}{2}; -\lambda^2\gamma^2) + \mu_i^2\gamma^3}{3} \quad (16)$$

where ${}_2F_1$ is the hypergeometric function of the given arguments with the real part being used in this evaluation. It is interesting to note that an analytic continuation formula from [1] has been used to calculate the hypergeometric function in lieu of the more classic Taylor series method.

Similarly for the Cross flow, in a 3D circular tube, we have :

$$Q = \frac{\pi(2L)^3 I}{(\Delta P)^3} \quad (17)$$

and :

$$\Big[\mu_i + \frac{\delta}{1+\lambda^n\gamma^n}\Big]\gamma = \frac{R\Delta P}{2L} \quad (18)$$

if we pose $f = \lambda^n\gamma^n$ and $g = 1 + f$ then :

$$I = I1 + I2 \quad (19)$$

with :

$$I1 = \gamma^4 \frac{2\delta^3\Big[-n(2f^2+5f+3) + 4g^2 + 2n^2\Big] + 12n\delta^2\mu_i g(n-g) + 12n^2\delta\mu_i^2 g^2 + 3n^2\mu_i^3 g^3}{12n^2g^3} \quad (20)$$

$$I2 = \gamma^4{}_2F_1(1, \frac{4}{n}; \frac{n+4}{n}; -f)\frac{\delta^3(n^2-6n+8) + 3n\delta^2\mu_i(n-4) + 3m^2\delta\mu_i^2}{12n^2} \quad (21)$$

Finally for the 2D Cross flow case, we have :

$$Q = \frac{2\pi(L)^2 I}{(\Delta P)^2} \quad (22)$$

and :

$$\Big[\mu_i + \frac{\delta}{1+\lambda^n\gamma^n}\Big]\gamma = \frac{B\Delta P}{L} \quad (23)$$

$$I = \gamma^3 \frac{3\delta^2(n-g) - (\delta^2(n-3) + 2n\delta\mu_i)g^2{}_2F_1(1, \frac{3}{n}; 1+\frac{3}{n}; -f) + 6n\delta\mu_i g + 2n\mu_i^2 g^2}{6ng^2} \quad (24)$$

For the numerical analysis, the geometrical parameters will be the same as those chosen in [2]. For the Carreau 3D case, $L = 0.5$ and $R = 0.02$. For the 2D case, $L = 1.3$ and $B = 0.012$. For the Cross 3D case, $L = 0.95$ and $R = 0.008$. For the 2D case, $L = 0.75$ and $B = 0.005$

3.3 Model Description

The mesh size will be chosen in the various cases so as to have at least 20 elements through the thickness in each case. The Parameters chosen are again consistent with the models shown in [2].

	2D model	3D model
Surface Element size	0.0012	0.05
Volume Nodes	26000	36397
Volume Elements	50000	200546
Anisotropic Elements added to the Boundary Layer	2	2

Table 2: Test Case Mesh Information - Carreau fluid

	2D model	3D model
Surface Element size	0.0005	0.0008
Volume Nodes	40527	514112
Volume Elements	78000	2954978
Anisotropic Elements added to the Boundary Layer	2	2

Table 3: Test Case Mesh Information - Cross fluid

	2D model	3D model
ρ	1	1
n	0.75	0.9
μ_0	0.17	0.08
μ_i	0.009	0.001
λ	2.5	2.0

Table 4: Test Case Parameters - Carreau fluid

	2D model	3D model
ρ	1	1
n	0.45	0.83
μ_0	0.08	0.22
μ_i	0.003	0.033
λ	0.75	6.65

Table 5: Test Case Parameters - Cross fluid

4 Test Case Results

4.1 Test Case observations

In can be seen in Table (6), Table (7), Table (8) and Table (9) that the numerical results agree well with the analytical solution for both the Carreau and Cross flows for the different pressure gradients applied between the inlet and the outlet.

ΔP	Analytical	Simulation
500	$0.1396e^{-2}$	$0.1360e^{-2}$
1000	$0.3009e^{-2}$	$0.2931e^{-2}$
1500	$0.4716e^{-2}$	$0.4593e^{-2}$
2000	$0.6487e^{-2}$	$0.6316e^{-2}$

Table 6: Test Case Mesh Results - 3D Carreau fluid

ΔP	Analytical	Simulation
500	$0.8083e^{-2}$	$0.8279e^{-2}$
1000	$1.932e^{-2}$	$1.989e^{-2}$
1500	$3.204e^{-2}$	$3.302e^{-2}$
2000	$4.577e^{-2}$	$4.724e^{-2}$

Table 7: Test Case Mesh Results - 2D Carreau fluid

ΔP	Analytical	Simulation
500	$0.2447e^{-4}$	$0.2387e^{-7}$
1000	$0.4996e^{-4}$	$0.4876e^{-4}$
1500	$0.7551e^{-4}$	$0.7370e^{-4}$
2000	$1.011e^{-4}$	$0.9866e^{-4}$

Table 8: Test Case Mesh Results - 3D Cross fluid

ΔP	Analytical	Simulation
500	$0.5747e^{-2}$	$0.5848e^{-2}$
1000	$1.481e^{-2}$	$1.5091e^{-2}$
1500	$2.529e^{-2}$	$2.580e^{-2}$
2000	$3.665e^{-2}$	$3.734e^{-2}$

Table 9: Test Case Mesh Results - 2D Cross fluid

References

[1] J. Pearson, *Computation of hypergeometric functions*, Master's thesis, Worcester College, 2009.

[2] T. Sochi, *Analytical solutions for the flow of carreau and cross fluids in circular pipes and thin slits*, Rheologica Acta, 54 (2015), pp. 745–756.