COMMUNICATIONS IN NUMERICAL METHODS IN ENGINEERING
Commun. Numer. Meth. Engng 2006; **22**:77–82

Iterative solution technique in selective mass scaling

Lars Olovsson[1,*,†] and Kjell Simonsson[2,‡]

[1]*Swedish Defence Research Agency, SE-147 25 Tumba, Sweden*
[2]*Division of Solid Mechanics, Linköping University, SE-581 83 Linköping, Sweden*

SUMMARY

Selective mass scaling in explicit finite element analyses is a method to increase the critical time step by modifying the mass matrix while at the same time leaving the rigid body translational behaviour unaffected. In an earlier work it has been suggested to calculate the global acceleration vector as $\mathbf{a}=\bar{\mathbf{M}}^{-1}\mathbf{f}$ where $\bar{\mathbf{M}}$ is the selectively scaled global mass matrix and where $\mathbf{f}$ is the global node force vector. However, dealing with large models, the mass matrix inversion is computationally costly and the storage of $\bar{\mathbf{M}}^{-1}$ requires a considerable amount of memory space. In this context it is here shown that an iterative solution technique, where $\bar{\mathbf{M}}\mathbf{a}=\mathbf{f}$ is solved every time step, can be an attractive alternative to inverting the mass matrix. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: selective mass scaling; finite element; explicit time integration; conjugate gradient

1. INTRODUCTION

Selective mass scaling was originally introduced by Olovsson *et al.* [1], as a method to increase the critical time step in explicit finite element analyses[§] of thin walled structures modelled with tri-linear solid elements. The idea was to lower the highest eigenfrequencies by averaging the accelerations of close nodes. A more formal and general procedure, applicable to all kinds of displacement-based elements, was then presented by Olovsson *et al.* [3]. By allowing the mass matrix to be modified under the constraint that the rigid body translational behaviour of each element in the body is to be preserved, it was shown that the highest eigenfrequencies could be significantly decreased while leaving the lower eigenfrequency domain relatively unaffected. In this paper, focus will be placed on this latter approach and, more specifically, on its efficiency when using an iterative solution technique. It is to be noted that

---

[*] Correspondence to: Lars Olovsson, Swedish Defence Research Agency, SE-147 25 Tumba, Sweden.
[†] E-mail: lars.olovsson@foi.se
[‡] E-mail: kjesi@ikp.liu.se

[§] For a comprehensive treatment of non-linear FEM we refer Reference [2].

Received 1 March 2005
Revised 7 June 2005
Accepted 22 July 2005

even though the discussion and numerical example will be given in the context of tri-linear solid elements, the obtained conclusions are of general applicability.

The selective scaling of the mass matrix may be done in a number of different ways, see Reference [3]. One simple method which has been found to work well is to define the global mass matrix $\bar{\mathbf{M}}$ according to

$$\bar{\mathbf{M}} = \mathrm{A}_e \bar{\mathbf{m}}^e, \quad \bar{\mathbf{m}}^e = \mathbf{m}^e + \boldsymbol{\lambda}^e \tag{1}$$

where A_e represents the assembly operation, and where $\mathbf{m}^e$ and $\boldsymbol{\lambda}^e$ denote the ordinary lumped element mass matrix and an augmentation matrix, respectively.

$$\mathbf{m}^e = \frac{m^e}{8}\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & & \\ \vdots & & \ddots & \\ 0 & & & 1 \end{bmatrix} \tag{2}$$

$$\boldsymbol{\lambda}^e = \begin{bmatrix} \boldsymbol{\lambda}^e_{8\times 8} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\lambda}^e_{8\times 8} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\lambda}^e_{8\times 8} \end{bmatrix} \tag{3}$$

$$\boldsymbol{\lambda}^e_{8\times 8} = \frac{\beta m^e}{56}\begin{bmatrix} 7 & -1 & \dots & -1 \\ -1 & 7 & & \\ \vdots & & \ddots & \\ -1 & & & 7 \end{bmatrix} \tag{4}$$

As can be seen, the row and column sums of $\boldsymbol{\lambda}^e_{8\times 8}$ are all zero, implying that the rigid body translational behaviour predicted by $\bar{\mathbf{m}}^e$ and $\mathbf{m}^e$ will be the same. Furthermore, the constant β is defined such that the value $\beta = 1$ doubles the diagonal terms of the original mass matrix. Numerical tests indicate that the critical time step size roughly increases by a factor $\sqrt{1+\beta}$.

Having assembled $\bar{\mathbf{M}}$ and calculated its inverse, the global node acceleration vector $\mathbf{a}$ can be calculated from the global node force vector $\mathbf{f}$, according to

$$\mathbf{a} = \bar{\mathbf{M}}^{-1}\mathbf{f} \tag{5}$$

However, dealing with large models, the mass matrix inversion is computationally costly and the storage of $\bar{\mathbf{M}}^{-1}$ requires a considerable amount of memory space. Even though $\bar{\mathbf{M}}$ is sparse, $\bar{\mathbf{M}}^{-1}$ is not. With this as a background it seems reasonable to avoid inverting $\bar{\mathbf{M}}$ and instead solve the sparse equation system $\bar{\mathbf{M}}\mathbf{a} = \mathbf{f}$ every time step. This would also open up for the possibility of defining a mass matrix that is not constant in time, which is very important in situations of element erosion and h-adaptivity where the node and element connectivities may change in time.

2. ITERATIVE SOLUTION TO $\bar{\mathbf{M}}\mathbf{a} = \mathbf{f}$

Due to the sparsity of the mass matrix $\bar{\mathbf{M}}$, the conjugate gradient method (CGM), see, e.g. Reference [4], is an attractive choice for solving the equation system $\bar{\mathbf{M}}\mathbf{a} = \mathbf{f}$. In addition, since the acceleration vector found in one time step generally will serve as a good initial guess for the next time step, the method also has the potential of being quite effective. However, the convergence rate of the CGM depends on the condition number of $\bar{\mathbf{M}}$. The number of iterations required to meet a given convergence criterion is actually proportional to the square root of the condition number.

Hence, the method is less efficient for situations with large differences in node masses. Further, the selective mass scaling procedure itself increases the condition number by roughly a factor $1 + 2\beta$. Consequently, the necessary number of iterations becomes proportional to $\sqrt{1 + 2\beta}$. Depending on the cost for each iteration one may reach a limit where the physics allows for large scale factors but where the nature of the CGM makes them uninteresting, unless some pre-conditioning technique is applied.

A convergence criterion is needed to determine when to terminate the iteration process. In this work the solution was assumed good enough when

$$\frac{\|\bar{\mathbf{M}}\mathbf{a}^n - \mathbf{f}\|_2}{\|\mathbf{f}\|_2} \leqslant 10^{-3} \tag{6}$$

where $\mathbf{a}^n$ denotes the approximative solution after n iterations.

3. APPLICATION EXAMPLE

The proposed iterative solution technique has been tested for an impact situation where a steel tube hits a rigid inclined plane at a velocity of 100 m/s, see Figure 1.

Even though the selective mass scaling technique has its greatest potential in the simulation of quasi-static processes, the chosen example will show that it is also applicable at situations

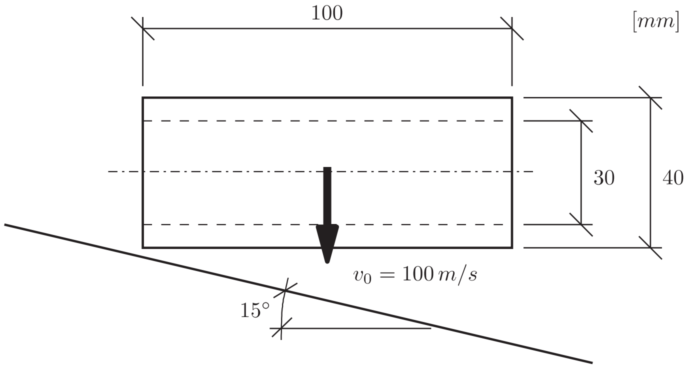

Figure 1. A steel tube impacting a rigid inclined surface.

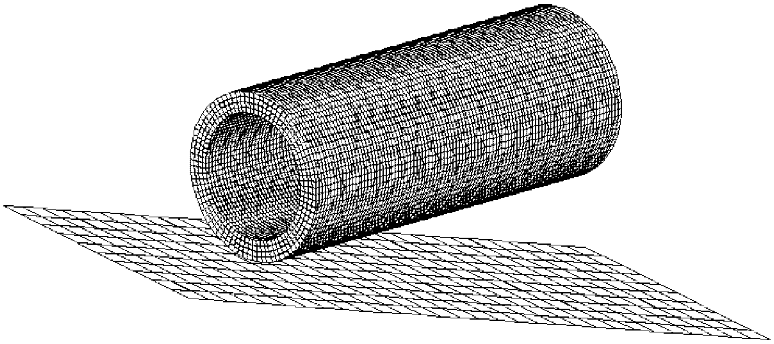

Figure 2. Finite element discretization of a steel tube that is about to impact a rigid plane.

in which inertia effects are of utmost importance. The tube was discretized with 32 000 selectively reduced integrated solid elements according to Figure 2.

The Young's modulus, Poisson's ratio and the density were set to $E = 207\,\text{GPa}$, $\nu = 0.3$ and $\rho = 7800\,\text{kg/m}^3$, respectively. A von Mises yield criterion was assumed and the yield stress σ_y was defined as a function of the effective plastic strain $\varepsilon_{\text{eff}}^{\text{p}}$ according to

$$\sigma_y = 792 + 510[\varepsilon_{\text{eff}}^{\text{p}}]^{0.26} \quad [\text{MPa}] \tag{7}$$

Four simulations with different mass matrices were carried out, using the explicit finite element code KRYP, c.f. Reference [5]. The tested cases were; no mass scaling, regular mass scaling with a doubled density and selective mass scaling with $\beta = 10$ and 50. The simulations were terminated after 500 μs, as the tube was no longer in contact with the inclined plane. Figure 3 shows the final shape of the deformed tube and the resulting contact forces are presented in Figure 4. Computational speed-up data at the different levels of mass scaling is given in Table I.

From the results it can be seen that selective mass scaling, in contrast to regular mass scaling, can be used to cut the simulation time, not only in quasi-static analyses, but also in dynamic ones. Furthermore, the maximum acceptable level of β will depend on the accuracy requirement.

4. DISCUSSION

This work has shown that a computational speed-up can be obtained when applying the CGM to solve for the acceleration vector in selectively mass scaled systems. The magnitude of the speed-up depends both on the number of operations required to solve $\bar{\mathbf{M}}\mathbf{a} = \mathbf{f}$ and on the total number of operations needed to complete one time step. Hence, it naturally becomes both program and problem dependent. Since the convergence rate is slower for badly conditioned systems, direct solution methods might be more efficient at large mass scaling factors β.

The given application example also shows that selective mass scaling can be applied when analysing dynamical processes, where inertia effects are of utmost importance. However, the

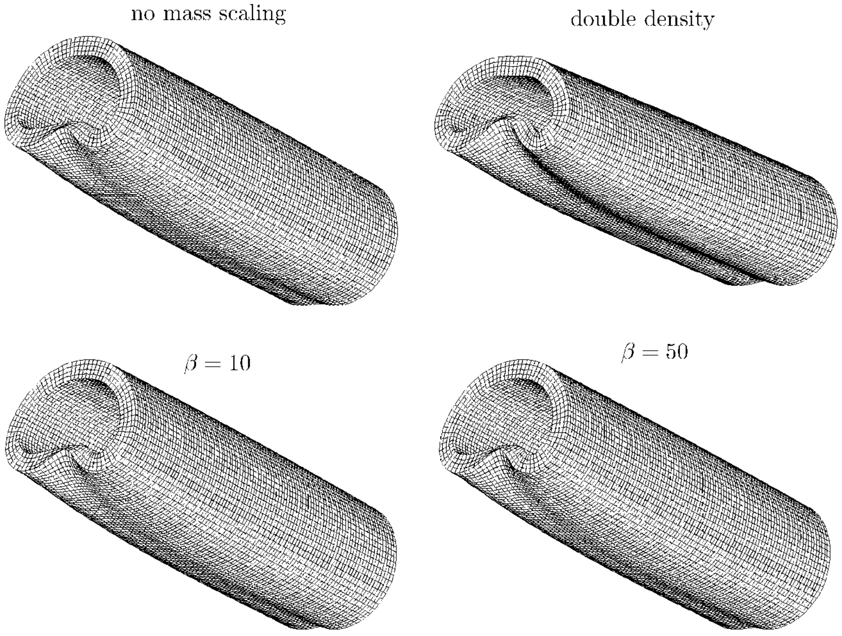

Figure 3. Steel tube after impact at different levels of mass scaling.

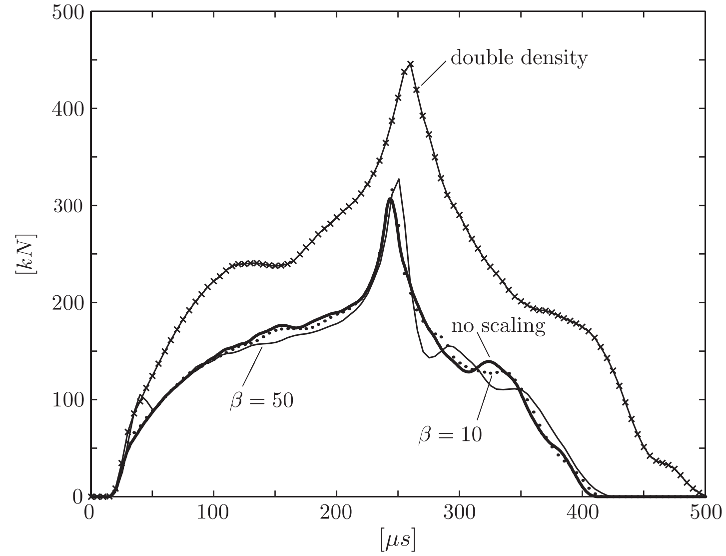

Figure 4. Resultant contact force at different levels of mass scaling.

Table I. Speed-up, total number of time steps and average number of conjugate gradient iterations per time step at different levels of mass scaling.

Case	Speed-up	Number of time steps	Avg no iterations
No mass scaling	1	4268	—
Double density	1.2	3149	—
$\beta = 10$	2.0	1346	13.1
$\beta = 50$	3.4	621	24.0

proposed method is not suitable in situations where the required resolution of the element grid is controlled by the need to accurately represent high frequencies. This is generally the case when dealing with the propagation of shock waves and nearly discontinuous stress waves.

Further, using a penalty-based contact formulation, increased time steps generally magnify the numerical noise in the contact force distribution. Hence, it is important to stress that the proposed method is not attractive whenever the time step size is limited by the need for accuracy in the contact treatment.

Due to its simplicity, the iterative solution technique is attractive in situations where only small parts of the model need to be mass scaled. Un-coupled diagonal terms in the mass matrix can easily be excluded from the iterative solution process.

REFERENCES

1. Olovsson L, Unosson M, Simonsson K. Selective mass scaling for thin walled structures modeled with tri-linear solid elements. *Computational Mechanics* 2004; **34**(2):134–136.
2. Belytschko T, Liu WK, Moran B. *Nonlinear Finite Elements for Continua and Structures*. Wiley: New York, 2000.
3. Olovsson L, Simonsson K, Unosson M. Selective mass scaling for explicit finite element analysis. *International Journal for Numerical Methods in Engineering* 2005; **63**(10):1496–1505.
4. Nash SG, Sofer A. *Linear and Nonlinear Programming*. McGraw-Hill: New York, 1996.
5. Olovsson L. *KRYP, A Finite Element Tool for Crystal Plasticity Analyses*. Swedish Defence Research Agency: Tumba, 2002.